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Closed characteristics on singular energy levels of second-order Lagrangian systems

William D. Kalies *, Mark Wess

Department of Mathematical Sciences, Florida Atlantic University, Boca Raton, FL 33431, USA

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Abstract

We provide a lower bound on the number of closed characteristics on singular energy levels of second-order Lagrangian systems in the presence of saddle-focus equilibria. The hypotheses on the Lagrangian are mild, and the bound is given in terms of the number of saddle-foci and a potential function determined by the Lagrangian. The method of proof is variational, combining techniques to minimize near a saddle-focus and an analog of the method of broken geodesics.

Keywords: Second-order Lagrangian; Saddle-focus; Closed characteristic; Minimization

1. Introduction

Second-order Lagrangian systems are used as models of various phenomena in nonlinear elasticity, optics, and mechanics. These conservative, fourth-order differential equations are obtained variationally as the Euler–Lagrange equations of an action functional which depends on the second derivative of the state variable u as well as its lower derivatives. One important family of such differential equations is $u'''' - \beta u'' + f(u) = 0$, which is known as the Swift–Hohenberg (SH) equation for $\beta \leqslant 0$ and the extended Fisher–Kolmogorov (eFK) equation for $\beta > 0$. These equations have been studied in a variety of contexts, see [7,8,10,14,21] and the references therein.

In particular, the existence of multibump heteroclinic, homoclinic, and periodic solutions has been extensively studied in this family of systems, e.g. [2–5,9,10,12,18,20,22]. Of particular interest has been the existence of many heteroclinic and homoclinic solutions between saddle-

* Corresponding author.
E-mail address: wkalies@fau.edu (W.D. Kalies).

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focus equilibria. The presence of complicated, indeed chaotic, dynamics near saddle-foci has been known since Shil'nikov [17] and in fourth-order, conservative equations since Devaney [6]. A variational approach to finding solutions near saddle-foci in second-order Lagrangian systems is developed in [9,10] which establishes the existence of many solutions that cannot be detected by the dynamical systems techniques and previously known variational methods. The main drawback of the method in [9,10] is its originally narrow applicability to systems of the eFK type for which the action functional is bounded below. The main goal of this paper is to demonstrate that a suitably modified version of this variational technique can be applied to general class of second-order Lagrangian systems under relatively mild hypotheses. We also make use of an extension of this technique by Bonheure [4].

In [11], the method of broken geodesics is applied to a general class of second-order Lagrangian systems to give a lower bound on the number of periodic orbits, or closed characteristics, on regular energy manifolds. The bound is given in terms of the topology of the manifold which can be determined from a potential function, see [1,11]. While there are results which provide lower bounds on the number of closed characteristics on classes of manifolds with certain geometric properties, e.g. [15,23,24], energy manifolds of second-order Lagrangian systems are noncompact and do not necessarily satisfy the typically required properties, see [1]. For results concerning mechanical Hamiltonian systems see [13].

This paper extends the two variational methods in [9,10] and [11]. For a general class of systems, we prove a lower bound on the number of closed characteristics on singular energy level sets of a second-order Lagrangian systems with saddle-focus equilibria, under relatively mild hypotheses.

1.1. Second-order Lagrangian systems

Given a second-order Lagrangian density of the form $L = L(u, u', u'')$ and taking the first variation of the associated action functional $J[u] = \int_I L(u, u', u'')\,dt$ for some interval I gives a necessary condition for extremizing this action, the Euler–Lagrange equation given by

$$\frac{d^2}{dt^2}\partial_{u''}L - \frac{d}{dt}\partial_{u'}L + \partial_u L = 0. \tag{1}$$

The specific Lagrangians that we consider in this paper satisfy the hypothesis

(H1) $L(u, u', u'') = \frac{1}{2}|u''|^2 + K(u, u')$.

Under this hypothesis, a second-order Lagrangian system gives rise to a flow in $\mathbb{R}^4$ with Hamiltonian

$$H(u, u', u'', u''') = \left(\partial_{u'}L - \frac{d}{dt}\partial_{u''}L\right)u' + (\partial_{u''}L)u'' - L = (\partial_{u'}K - u''')u' + \frac{1}{2}|u''|^2 - K.$$

Introducing symplectic coordinates $(u, u', p_u, p_{u'})$, where $p_u = \partial_{u'}L - \frac{d}{dt}\partial_{u''}L = \partial_{u'}K - u'''$ and $p_{u'} = \partial_{u''}L = u''$, the Hamiltonian becomes $H = p_u u' + L^*(u, u', p_{u'})$ where L^* is the Legendre transform of L with respect to u''. The level sets $M_E = \{(u, u', u'', u''') \mid H(u, u', u'', u''') = E\}$ are invariant under the flow of Eq. (1). If $\nabla H \neq 0$ on M_E, then E is a regular value and M_E is a smooth, noncompact 3-manifold without boundary. If $\nabla H = 0$ for some critical point in M_E, then M_E is singular at such points. Computing

$$\nabla H = \left[\partial^2_{uu'} K(u,u')u' - \partial_u K(u,u'),\ \partial^2_{u'} K(u,u')u' - u''',\ u'',\ -u'\right]$$

implies that critical points of H have the form $(u_*, 0, 0, 0)$ with $\partial_u K(u_*, 0) = 0$ and such points coincide with equilibrium points of (1).

As described in the appendix of [19], the nondegenerate equilibrium points of the Euler–Lagrange equation (1) can be classified by the signs of $\partial^2_u K(u_*, 0)$ and $\partial^2_{u'} K(u_*, 0)$.

Lemma 1.1. *(See [19].) Let u_* be an equilibrium point of* (1) *with $\alpha = \partial^2_u K(u_*, 0)$ and $\beta = \partial^2_{u'} K(u_*, 0)$.*

(i) *If $\alpha < 0$, then u_* is a saddle-center.*
(ii) *If $\alpha > 0$, then u_* is a saddle-focus, center, or real saddle, depending on the value of β. In particular, if $\beta^2 < 4\alpha$, then u_* is a saddle-focus.*

In this paper we restrict ourselves to the saddle-focus case. Moreover, the techniques used in this paper require two additional hypotheses on the Lagrangian. The first is a growth condition, and the second is used to bound the action functional from below in certain spaces, namely

(H2) $|\partial_u K| \leqslant C(|u|)(1 + |u'|^\gamma)$, $|\partial^2_{uu'} K| \leqslant C(|u|)(1 + |u'|^{\gamma-1})$, $|\partial^2_{u'} K| \leqslant C(|u|)(1 + |u'|^{\gamma-2})$,
(H3) $K(u, u') \geqslant -C(|u|)(1 + |u'|^\gamma)$

for some $\gamma < 4$ and some locally bounded function $C(|u|) > 0$.

1.2. Closed characteristics

In [11], it is shown that for regular energy manifolds, the number of closed characteristics can be bounded below by the second Betti number of M_E, which in turn can be computed from superlevel sets of the potential function $L(u, 0, 0)$, i.e. $\dim H_2(M_E)$ is the number of compact intervals on which $L(u, 0, 0) + E \geqslant 0$. In the singular case, a lower bound can be given in terms of the number of such intervals as well as the number of saddle-focus equilibria in the intervals.

Theorem 1.2. *Suppose L is a C^2 Lagrangian satisfying hypotheses* (H1)–(H3). *Let M_E be a singular energy level set containing only saddle-focus equilibria of the Euler–Lagrange equation* (1). *Let n be the number of compact intervals on which $L(u, 0, 0) + E \geqslant 0$, and let e be the number of saddle-focus equilibrium points in M_E. Then the number of closed characteristics of M_E is bounded below by $n + 2e$.*

The proof of Theorem 1.2 uses the method of broken geodesics explained in Section 2. Key to this method is the existence of monotone laps as proved in Section 4, beginning with some preliminary results in Section 3.

2. Existence of closed characteristics

We now give a formal description of the existence of closed characteristics by the method of broken geodesics.

2.1. Broken geodesics

Definition 2.1. For $u_1 < u_2$, an *increasing lap* from u_1 to u_2 is a solution to the Euler–Lagrange equation

$$u'''' - \partial^2_{u'} K(u, u')u'' - \partial^2_{uu'} K(u, u')u' + \partial_u K(u, u') = 0 \tag{2}$$

satisfying the boundary conditions $u(0) = u_1$, $u'(0) = 0$, $u(T) = u_2$, $u'(T) = 0$ and $u'(t) > 0$ for $0 < t < T$ with a similar definition of *decreasing lap*. A *simple closed characteristic of type* (u_1, u_2) is a periodic solution to (2) for which each period is composed of a single increasing lap from u_1 to u_2 and a single decreasing lap from u_2 to u_1.

At times where $u' = 0$, solutions to (2) satisfy $\frac{1}{2}|u''|^2 - K(u, 0) = E$. Let N be this level set in the (u, u'')-plane. Every simple closed characteristic intersects N exactly twice. Let πN be the projection of N to the u-axis. The set $N \cap \{(u, u'') \mid u'' > 0\}$ is a graph over πN in the (u, u'')-plane as is $N \cap \{(u, u'') \mid u'' < 0\}$. In particular $\pi N = \{u \in \mathbb{R} \mid L(u, 0, 0) + E \geqslant 0\}$, and we refer to the connected components of πN as *interval components*.

Consider a compact interval component $I = [u_-, u_+]$ which contains a point u_* in its interior corresponding to a saddle-focus equilibrium of (2). Initially we assume that u_* is the only equilibrium in $[u_-, u_+]$. Let $B = \{(u_1, u_2) \in I \times I \mid u_1 < u_2\}$. Our goal is to find points $(u_1, u_2) \in B$ for which there exists a simple closed characteristic which is the concatenation of an increasing and a decreasing lap. For an increasing lap l_+ from u_1 to u_2 we let $p^+_{u_1}$ and $p^+_{u_2}$ be the p_u-values at the concatenation points, and also for a decreasing lap l_- we let $p^-_{u_1}$ and $p^-_{u_2}$ be the corresponding p_u-values, see Fig. 1. If u is the concatenation of l_+ and l_-, then necessary conditions for u to be a solution of (2) are $p^+_{u_1} - p^-_{u_1} = 0$ and $p^+_{u_2} - p^-_{u_2} = 0$. Since l_+ and l_- are solutions to (2), their intersection with N determines the values of $u'' = p_{u'}$ uniquely from u_1 and u_2, and we denote these values by $p_{u'}(u_1)$ and $p_{u'}(u_2)$. Thus the necessary compatibility conditions on the p_u-values are also sufficient.

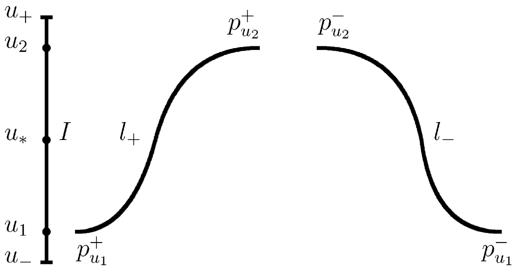

Fig. 1. A simple closed characteristic is a concatenation of an increasing lap l_+ with a decreasing lap l_-. The $p_{u'}$-values at the endpoints of each lap are determined by the minimum and maximum values u_1 and u_2, but the p_u-values are not, which gives a necessary and sufficient compatibility condition for the existence of a simple closed characteristic, $p^+_{u_2} = p^-_{u_2}$ and $p^+_{u_1} = p^-_{u_1}$.

Let $u(t)$ be the solution to (2) with initial conditions $u(0) = u_1$, $u'(0) = 0$, $u''(0) = p_{u'}(u_1) > 0$, and $u'''(0) = \partial_{u'} K(u_1, 0) - p^+_{u_1}$ which depend only on $(u_1, p^+_{u_1}) \in I \times \mathbb{R}$. Such a solution is initially monotonically increasing. Define P_+ to be all values $(u_1, p^+_{u_1}) \in I \times \mathbb{R}$ for which there is a time $t_1 > 0$ such that $u'(t_1) = 0$ with $u(t_1) \in I$ and $u'(t) > 0$ for all $0 < t < t_1$. Similarly, solutions with initial conditions $u(0) = u_2$, $u'(0) = 0$, $u''(0) = p_{u'}(u_2) < 0$, and $u'''(0) = \partial_{u'} K(u_1, 0) - p^-_{u_2}$ are initially monotonically decreasing, and we define P_- to be all the values $(u_2, p^-_{u_2}) \in I \times \mathbb{R}$ for which there is a time $t_2 > 0$ such that $u'(t_2) = 0$ with $u(t_2) \in I$ and $u'(t) < 0$ for all $0 < t < t_2$. Further define $f_+ : P_+ \to I$, $g_+ : P_+ \to \mathbb{R}$ by $f_+(u_1, p^+_{u_1}) = u(t_1)$ and $g_+(u_1, p^+_{u_1}) = p_u(t_1)$ and $f_-, g_- : P_- \to \mathbb{R}$ by $f_-(u_2, p^-_{u_2}) = u(t_2)$ and $g_-(u_2, p^-_{u_2}) = p_u(t_2)$. By continuity of solutions of (2) with respect to initial conditions, all of these functions are continuous on any open subset of their domains. Note that the points $(u_*, 0)$ are excluded from the domains $P_\pm$ by definition.

Let

$$\mathcal{L}_+ = \{(u, f_+(u, p_u), p_u, g_+(u, p_u)) \mid (u, p_u) \in P_+\} \quad \text{and}$$
$$\mathcal{L}_- = \{(f_-(u, p_u), u, g_-(u, p_u), p_u) \mid (u, p_u) \in P_-\}$$

which are two-dimensional graphs over the (u, p_u)-plane with domains P_+ and P_-, respectively. Points of intersection of $\mathcal{L}_+$ and $\mathcal{L}_-$ correspond to broken geodesics where the p_u-values are compatible and hence give simple closed characteristics.

Let $\pi\mathcal{L}_\pm$ be the projection of the graphs $\mathcal{L}_+$ and $\mathcal{L}_-$ onto the first two coordinates. First we show that $\pi\mathcal{L}_\pm$ cover all of B, except possibly points at u_*, that is, there exists at least one monotone lap for each pair $(u_1, u_2) \in B \setminus \{(u, u_*)$ and $(u_*, u) \mid u \in I\}$. In fact the methods of Section 4 can be used to establish the existence of laps which terminate at u_* as well, but they are not required for the main result.

Theorem 2.2. *If L satisfies hypotheses* (H1)–(H3) *and $u_* \in (u_-, u_+)$ is a saddle-focus with (u_-, u_+) containing no other equilibria, then for each pair $(u_1, u_2) \in B$ with $u_1, u_2 \neq u_*$ there exists an increasing lap from u_1 to u_2. The same is true for decreasing laps. In particular $B \setminus \{(u, u_*)$ and $(u_*, u) \mid u \in I\} \subset \pi\mathcal{L}_\pm$.*

Proof. The case $u_1 < u_* < u_2$ is proved in Theorem 4.15 of Section 4. The other cases are proved in Theorem 3.12 of [11]. □

The intersection $\mathcal{L}_+ \cap \mathcal{L}_-$ can be characterized using the maps

$$F_+(u_1, p_{u_1}, u_2, p_{u_2}) = [u_2 - f_+(u_1, p_{u_1}), p_{u_2} - g_+(u_1, p_{u_1})] \quad \text{and}$$
$$F_-(u_1, p_{u_1}, u_2, p_{u_2}) = [u_1 - f_-(u_2, p_{u_2}), p_{u_1} - g_-(u_2, p_{u_2})],$$

where the domain of F_+ is $P_+ \times \mathbb{R}^2$ and the domain of definition of F_- is $\mathbb{R}^2 \times P_-$. Define $F(u_1, p_{u_1}, u_2, p_{u_2}) = [F_+, F_-]$ on $P_+ \times P_-$. With this definition we have $F^{-1}(0, 0, 0, 0) = \mathcal{L}_+ \cap \mathcal{L}_-$.

To show that such intersection points exist, i.e. $F^{-1}(0, 0, 0, 0)$ is nonempty, we use the Brouwer degree. The degree is computed via a homotopy from the original Lagrangian L to a Lagrangian which satisfies the twist property originally defined in [19]. A brief explanation of the twist property and its implications on the existence of solutions is provided for completeness in Section 2.3. Crucial to this technique is the requirement that the projection of the intersection $\mathcal{L}_+ \cap \mathcal{L}_-$ onto B be contained in a compact subset of the interior of B. In Section 2.2, we prove a series of lemmas establishing the requisite properties of $\mathcal{L}_+ \cap \mathcal{L}_-$.

2.2. Properties of $\mathcal{L}_+ \cap \mathcal{L}_-$

In this section we work in a setting where we have a parametrized family of Lagrangians $L_\lambda(u, u', u'') = \frac{1}{2}|u''|^2 + K_\lambda(u, u')$ with the parameter $\lambda \in [0, 1]$ for which $K_\lambda(u, 0)$ is independent of λ, and hence the interval components and equilibrium points are independent of λ. We also assume that K_λ satisfies hypotheses (H2) and (H3) for each $\lambda \in [0, 1]$ with $\gamma, C(|u|)$ independent of λ.

Proposition 2.3. *Let $I = [u_-, u_+]$ be an interval component of πN_E. Then there exists $C(I) > 0$ such that if $\lambda \in [0, 1]$ and u is a simple closed characteristic for L_λ of type $(u_1, u_2) \in B$, then $|u'''(0)| \leqslant C(I)$. Moreover, if $s > 0$ is the time such that $u(s) = u_2$ then $|u'''(s)| \leqslant C(I)$.*

Proof. We bound $|u'''(0)|$, and a bound for $|u'''(s)|$ is obtained in an analogous way. We first show that for every compact subset $\widehat{B}$ of B there exists a bound $C(\widehat{B})$. Suppose there is not an upper bound on $u'''(0)$ for all increasing laps from u_1 to u_2 with $(u_1, u_2) \in \widehat{B}$ and $\lambda \in [0, 1]$. Then there exist sequences $\lambda_n \to \lambda_0$ in $[0, 1]$ and $(u_1^n, u_2^n) \to (u_1, u_2)$ in $\widehat{B}$ together with a sequence of increasing laps $u_n(t)$ for the Lagrangian L_{λ_n} from u_1^n and u_2^n with initial conditions $u_n(0) = u_1^n$, $u_n'(0) = 0$, $u_n''(0) = p_{u'}(u_1^n)$, and $u'''(0) = \xi_n$ where $\xi_n \to \infty$. Rescaling the independent variable by $\tau = \xi_n^{1/3} t$ with $w_n(\tau) = u_n(\xi_n^{-1/3}\tau)$, the initial value problem for (2) becomes

$$\ddddot{w}_n - \xi_n^{-2/3}\partial_2^2 K_{\lambda_n}\big(w_n, \xi_n^{1/3}\dot{w}_n\big)\ddot{w}_n - \xi_n^{-1}\partial_{12}^2 K_{\lambda_n}\big(w_n, \xi_n^{1/3}\dot{w}_n\big)\dot{w}_n$$
$$+ \xi_n^{-4/3}\partial_1 K_{\lambda_n}\big(w_n, \xi_n^{1/3}\dot{w}_n\big) = 0$$

with $w_n(0) = u_1^n$, $\dot{w}_n(0) = 0$, $\ddot{w}_n(0) = \xi_n^{-2/3} p_{u'}(u_1^n)$, and $\dddot{w}_n(0) = 1$, where $\dot{} = d/d\tau$.

Under hypothesis (H2) on the growth of $K_\lambda(u, u')$ in u', letting $n \to \infty$ and $\xi_n \to \infty$ we obtain the initial value problem $\ddddot{w} = 0$ with $w(0) = u_1$, $\dot{w}(0) = 0$, $\ddot{w}(0) = 0$, and $\dddot{w}(0) = 1$ which has solution $w(\tau) = \tau^3/6 + u_1$. Now using the continuous dependence of solutions with respect to initial conditions and perturbations of the vector field, we have

$$w_n(\tau) = u_n\big(\xi_n^{-1/3}\tau\big) \to \tau^3/6 + u_1 \quad \text{and} \quad \dot{w}_n(\tau) = \xi_n^{-1/3} u_n'\big(\xi_n^{-1/3}\tau\big) \to \tau^2/2$$

uniformly on compact intervals $[0, T]$ in τ. For all $n > 0$, let t_n be the time so that $u_n(t_n) = u_2^n$ and $u_n'(t_n) = 0$, and set $\tau_n = \xi_n^{1/3} t_n$ so that $w_n(\tau_n) = u_2^n$ and $\dot{w}_n(\tau_n) = 0$. Therefore $\tau_n \to [6(u_2 - u_1)]^{1/3}$ as $n \to \infty$ which implies $\dot{w}_n(\tau_n) \to [6(u_2 - u_1)]^{2/3}/2$ which contradicts $\dot{w}_n(\tau_n) = 0$ since $u_1 \neq u_2$. This implies there exists an upper bound $C(\widehat{B}) > 0$ such that $u'''(0) < C(\widehat{B})$ for all increasing laps from u_1 to u_2 with $(u_1, u_2) \in \widehat{B}$ and $\lambda \in [0, 1]$.

The above argument provides a uniform bound on $u'''(0)$ for all increasing laps from u_1 to u_2 on any compact subset $\widehat{B}$ in B and $\lambda \in [0, 1]$. If there were no such bound on all of B, then there exist sequences $\lambda_n \to \lambda_0$ and $(u_1^n, u_2^n) \to (\hat{u}, \hat{u})$ such that $u_n'''(0) = \xi_n \to \infty$ and $\hat{u} \in I$. Rescaling as before, $\tau = \xi_n^{1/3} t$, we obtain the limiting initial value problem $\ddddot{w} = 0$ with $w(0) = u_1$,

$\dot{w}(0)=0$, $\ddot{w}(0)=0$, and $\dddot{w}(0)=1$ which has solution $w(\tau)=\tau^3/6+\hat{u}$. Let t_n be defined as before so that $u_n(t_n)=u_2^n$, $u_n'(t_n)=0$, and $\tau_n=\xi_n^{1/3}t_n$. Then $\tau_n\to 0$ and hence $t_n=\xi_n^{-1/3}\tau_n\to 0$. By the Mean Value Theorem there must be a time $s_n\in[0,t_n]$ such that $u_n''(s_n)=0$ and a time $r_n\in[0,s_n]$ such that $u_n'''(r_n)=-p_{u'}(u_1)/s_n\leqslant 0$. Thus $\sigma_n=\xi_n^{1/3}r_n\to 0$ since $0\leqslant\sigma_n\leqslant\tau_n\to 0$, which implies $\ddot{w}_n(\sigma_n)=\xi_n^{-1}u_n'''(r_n)\leqslant 0$ but $\ddot{w}_n(\sigma_n)\to 1$ as $n\to\infty$, a contradiction. Therefore there exists $C>0$ such that $u'''(0)\leqslant C$ for any lap from u_1 to u_2 for $(u_1,u_2)\in B$ and $\lambda\in[0,1]$.

We now show that there is a lower bound. Following the same arguments as above with the change of variable $\tau=-\xi_n^{1/3}t$ and $w_n(\tau)=u_n(-\xi_n^{-1/3}\tau)$ yields the initial value problem

$$\ddddot{w}_n-\xi_n^{-2/3}\partial_2^2K_{\lambda_n}\big(w_n,-\xi_n^{1/3}\dot{w}_n\big)\ddot{w}_n+\xi_n^{-1}\partial_{12}^2K_{\lambda_n}\big(w_n,-\xi_n^{1/3}\dot{w}_n\big)\dot{w}_n$$
$$+\xi_n^{-4/3}\partial_1K_{\lambda_n}\big(w_n,-\xi_n^{1/3}\dot{w}_n\big)=0.$$

For $n\to\infty$ and $-\xi_n\to\infty$, we obtain the initial value problem $\ddddot{w}=0$ with $w(0)=u_1$, $\dot{w}(0)=0$, $\ddot{w}(0)=0$, and $\dddot{w}(0)=-1$ which has solution $w(\tau)=-\tau^3/6+u_1$. Using the same arguments as before, we obtain a contradiction to the assumption that $u_n'''(0)\to-\infty$ for some sequence of laps from u_1^n to u_2^n, which implies a uniform lower bound on any compact subset of B.

To get a uniform lower bound on all of B we must restrict to laps which are part of a simple closed characteristic. Suppose there exist sequences $\lambda_n\to\lambda_0$ and (u_1^n,u_2^n) converging to $(\hat{u},\hat{u})$ with simple closed characteristics u_n of L_{λ_n} of type (u_1^n,u_2^n) satisfying $u_n'''(0)=-\xi_n\to-\infty$. Let s_n be the time such that $u_n(s_n)=u_2^n$ and $t_n>s_n$ be the first time such that $u(t_n)=u_1^n$. For $\tau_n=-\xi_n^{1/3}t_n$, we have $\tau_n\to[6(u_1^n-u_1)]^{1/3}\to 0$ as $n\to\infty$, and hence $t_n=-\xi_n^{-1/3}\tau_n\to 0$. By the Mean Value Theorem there exists a time $q_n\in[s_n,t_n]$ such that $u_n''(q_n)=0$ and a time $r_n\in[s_n,q_n]$ such that $u_n'''(r_n)=-p_{u'}(u_2)/(q_n-s_n)\geqslant 0$. Then $\sigma_n=-\xi_n^{1/3}r_n\to 0$ since $0\leqslant\sigma_n\leqslant\tau_n\to 0$, and $\ddot{w}_n(\sigma_n)=-\xi_n^{-1}u_n'''(r_n)\geqslant 0$ but $\ddot{w}_n(\sigma_n)\to -1$ as $n\to\infty$, a contradiction. □

In the following lemmas we assume that $I=[u_-,u_+]$ contains exactly one equilibrium point u_* for all $\lambda\in[0,1]$ and furthermore that $u_*\in(u_-,u_+)$ is a hyperbolic saddle-focus equilibrium for all $\lambda\in[0,1]$.

Lemma 2.4. *There exists $\epsilon>0$, independent of $\lambda\in[0,1]$, such that for the neighborhood of the diagonal $\Delta_\epsilon=\{(u_1,u_2)\mid u_2-u_1<\epsilon\}$ in B we have $\pi(\mathcal{L}_+^\lambda\cap\mathcal{L}_-^\lambda)\cap\Delta_\epsilon=\emptyset$.*

Proof. Suppose not. Then there exist sequences $\lambda_n\to\lambda_0$ in $[0,1]$ and $(u_1^n,u_2^n)\in B$ corresponding to simple closed characteristics $u_n(t)$ such that $(u_1^n,u_2^n)\to(\bar{u},\bar{u})$ as $n\to\infty$. Then $u_n(t)$ is the solution with initial conditions $u_n(0)=u_1^n$, $u_n'(0)=0$, $u_n''(0)=p_{u'}(u_1^n)$, and $u_n'''(0)=\xi_n$. By Proposition 2.3, we can assume $\xi_n\to\xi$. Let $u(t)$ be the solution with initial conditions $u(0)=\bar{u}$, $u'(0)=0$, $u''(0)=p_{u'}(\bar{u})=0$, and $u'''(0)=\xi$ and with $\lambda=\lambda_0$. By continuity of solutions with respect to initial conditions and parameters, and the periodicity of u_n, we conclude $u(t)=\bar{u}$ for all $t\geqslant 0$, which is a contradiction if $\bar{u}\neq u_*$ or $\xi\neq 0$. So suppose that $\bar{u}=u_*$ and that $\xi=0$. Since $(u_*,0,0,0)$ is a hyperbolic equilibrium, there exists an isolating neighborhood N of $(u_*,0,0,0)$ in $\mathbb{R}^4$. Let t_n be the period of u_n. If t_n is bounded, then for n sufficiently large, the periodic orbit $\{(u_n(t),u_n'(t),u_n''(t),u_n'''(t))\mid t\in[0,t_n]\}$ is contained in N, which is a contradiction. Suppose that t_n is unbounded, and without loss of generality suppose the increasing lap time is unbounded. Then for n large enough there exist times s_1 and s_2 such that u_n satisfies the hypotheses of Theorem 3.3 on $[s_1, s_2]$ which implies that u_n oscillates around u_*, contradicting the monotonicity of u_n. $\square$

Lemma 2.5. *Let $u_n(t)$ be a sequence of increasing laps from u_1^n to u_2^n with initial conditions $u_n(0) = u_1^n$, $u_n'(0) = 0$, $u_n''(0) = p_{u'}(u_1^n)$, and $u_n'''(0) = \xi_n$ such that $u_1^n \to u_1$, $u_2^n \to u_2$, $\xi_n \to \xi$, and $\lambda_n \to \lambda_0$. Let t_n be the time such that $u_n(t_n) = u_2^n$, and let $u(t)$ be the solution with initial conditions $u(0) = u_1$, $u'(0) = 0$, $u''(0) = p_{u'}(u_1)$, $u'''(0) = \xi$, and $\lambda = \lambda_0$. If $t_n \to \infty$, then the omega-limit set $\omega(\{(u_1, 0, p_{u'}(u_1), \xi)\}) = \{(\alpha, 0, 0, 0)\}$ for some $\alpha \in [u_1, u_2]$.*

Proof. Since $u_n(t)$ is increasing on $[0, t_n]$ to u_2^n which converges to u_2 as $n \to \infty$, we have that $u(t)$ is increasing on $[0, \infty)$ and $\lim_{t\to\infty} u(t) = \alpha$ with $u_1 \leqslant \alpha \leqslant u_2$. We first show that $\omega(\{(u_1, 0, p_{u'}(u_1), \xi)\})$ is nonempty.

Case 1. There exists τ_1 such that $0 < u'(t) \leqslant 1$ for $t \geqslant \tau_1$.

Note that there cannot be a time τ_2 such that $|u''(t)| \geqslant 1$ for all $t \geqslant \tau_2$, otherwise $u(t)$ would eventually decrease or increase larger than α. Therefore either there exists a time τ_2 such that $|u''(t)| \leqslant 1$ for all $t \geqslant \tau_2$ or there exist strictly increasing sequences $r_k < s_k$ with $r_k, s_k \to \infty$ such that $u''(r_k) = u''(s_k) = 1$ for all $k > 0$ and $|u''(t)| \leqslant 1$ for all $t \in [r_k, s_k]$. In the former case, the Mean Value Theorem implies the existence of a sequence γ_k such that $\gamma_k \to \infty$ with $|u'''(\gamma_k)| \leqslant 1$. In the latter case, since $u''(s_k) - u''(r_k) = 0$, the Mean Value Theorem implies there exists $\gamma_k \in [r_k, s_k]$ such that $u'''(\gamma_k) = 0$. Therefore the sequence of points $(u(\gamma_k), u'(\gamma_k), u''(\gamma_k), u'''(\gamma_k))$ is contained in the compact set $[u_1, u_2] \times [0, 1] \times [-1, 1] \times [-1, 1]$ and hence has a convergent subsequence whose limit is an element of the ω-limit set.

Case 2. There does not exist a time τ_1 such that $0 < u'(t) \leqslant 1$ for $t \geqslant \tau_1$.

Note that there cannot be a time τ_2 such that $u'(t) \geqslant 1$ for all $t \geqslant \tau_2$, otherwise $u(t)$ would increase larger than α. Therefore, there exist strictly increasing sequences $r_k < s_k$ with $r_k, s_k \to \infty$ such that $u'(r_k) = u'(s_k) = 1$ for all $k > 0$ and $0 < u'(t) \leqslant 1$ for all $t \in [r_k, s_k]$. Since $u'(s_k) - u'(r_k) = 0$, there exists $\gamma_k \in [r_k, s_k]$ such that $u''(\gamma_k) = 0$. Therefore $u(\gamma_k)$, $u'(\gamma_k)$, and $u''(\gamma_k)$ are all bounded. Our goal then is to bound $u'''(\gamma_k)$ as well. Then the sequence of points $(u(\gamma_k), u'(\gamma_k), u''(\gamma_k), u'''(\gamma_k))$ is contained in a compact set which implies that the ω-limit set is nonempty.

If $u'(\gamma_k)$ is bounded away from zero, then we can use the Hamiltonian to bound $u'''(\gamma_k)$. Indeed, if there exists $U > 0$ such that $u'(\gamma_k) > U$ for all $k > 0$, then

$$u'''(\gamma_k) = \frac{\frac{1}{2}|u''(\gamma_k)|^2 - K_{\lambda_0}(u(\gamma_k), u'(\gamma_k)) - E}{u'(\gamma_k)} + \partial_{u'} K_{\lambda_0}\big(u(\gamma_k), u'(\gamma_k)\big)$$

which implies that $u'''(\gamma_k)$ is bounded.

Now we consider the final case that $u'(\gamma_k) \to 0$ as $k \to \infty$ and $u'''(\gamma_k)$ is unbounded. Suppose $u'''(\gamma_k) \to \infty$ as $k \to \infty$. Then we can form a sequence of solutions $v_k(t)$ with initial conditions $v_k(0) = u(\gamma_k)$, $v_k'(0) = u'(\gamma_k)$, $v_k''(0) = 0$, and $v_k'''(0) = \xi_k$. Using the rescaling arguments from the proof of Proposition 2.3 with $\tau = \xi_k^{1/3} t$ we arrive at the sequence of solutions $w_k(t)$ to the rescaled equations with $\lambda = \lambda_0$ and initial conditions $w_k(0) = u(\gamma_k)$, $\dot{w}_k(0) = u'(\gamma_k)/\xi^{1/3}$, $\ddot{w}_k(0) = 0$, and $\dddot{w}_k(0) = 1$. Note since $v_k(t) < \alpha$ for all t then $w_k(t) < \alpha$ for all t and all k.

Letting $k \to \infty$ we arrive at the differential equation $\ddddot{w}(0) = 0$ with initial conditions $w(0) = \alpha$, $\dot{w}(0) = 0$, $\ddot{w}(0) = 0$, and $\dddot{w}(0) = 1$ with solution $w(\tau) = \tau^3/6 + \alpha$. This implies that $w_k(\tau) \to \tau^3/6 + \alpha$ on the compact interval $[0, 1]$ which is a contradiction to $w_k(\tau) < \alpha$ for all τ and all k. Therefore $u'''(\gamma_k)$ must be bounded above. The argument for $u'''(\gamma_k) \to -\infty$ follows from similar arguments so that $u'''(\gamma_k)$ is also bounded below.

The above arguments show that $\omega(\{(u_1, 0, p_{u'}(u_1), \xi)\})$ is nonempty and that if $u(t)$ is the solution through $(u_1, 0, p_{u'}(u_1), \xi)$, then $\lim_{t\to\infty} u(t) = \alpha$. Suppose $(z_0, z_1, z_2, z_3) \in \omega(\{(u_1, 0, p_{u'}(u_1), \xi)\})$. Then there exist times $t_k \to \infty$ such that $(u(t_k), u'(t_k), u''(t_k), u'''(t_k)) \to (z_0, z_1, z_2, z_3)$, which implies that $z_0 = \alpha$. Now let $z(t)$ be the solution through (α, z_1, z_2, z_3). Since omega-limit sets are invariant, $(z(t), z'(t), z''(t), z'''(t)) \in \omega(\{(u_1, 0, p_{u'}(u_1), \xi)\})$ for all t. Hence $z(t) \equiv \alpha$, and $z_1 = z'(0) = 0$, $z_2 = z''(0) = 0$, and $z_3 = z'''(0) = 0$. Therefore $\omega(\{(u_1, 0, p_{u'}(u_1), \xi)\}) = \{(\alpha, 0, 0, 0)\}$. □

Lemma 2.6. *There exists $\epsilon > 0$, independent of $\lambda \in [0, 1]$, such that the neighborhood $\partial_\epsilon B = \{(u_1, u_2) \mid u_1 - u_- < \epsilon \text{ or } u_+ - u_2 < \epsilon\}$ of the left and top boundary of B satisfies $\pi(\mathcal{L}_+^\lambda \cap \mathcal{L}_-^\lambda) \cap \partial_\epsilon B = \emptyset$.*

Proof. Consider an increasing lap u at the initial point $u(0) = u_-$, $u'(0) = 0$, $u''(0) = 0$ and $u'''(0) = \xi$. Since u is increasing, $u'''(0) = \xi > 0$. Suppose that u is a simple closed characteristic. Then $u(0) = u_-$ is a minimum, but $\xi > 0$ implies $u(-t) < u_-$ for $t > 0$ and sufficiently small. Therefore there are no simple closed characteristics with minimum value $u(0) = u_-$. Similarly, there are no simple closed characteristics with maximum value u_+. Suppose there exist sequences $\lambda_n \to \lambda_0$ and $(u_1^n, u_2^n) \in B$ corresponding to simple closed characteristics u_n such that $(u_1^n, u_2^n) \to (u_-, \bar{u})$ as $n \to \infty$. Note that $\bar{u} > u_-$, since Lemma 2.4 prevents $\bar{u} = u_-$. Let $T_n > 0$ be the period of u_n, which has initial conditions $u_n(0) = u_1^n$, $u_n'(0) = 0$, $u_n''(0) = p_{u'}(u_1^n)$, and $u_n'''(0) = \xi_n \to \xi$ by Proposition 2.3. Let u be the solution with $\lambda = \lambda_0$ and $u(0) = u_-$, $u'(0) = 0$, $u''(0) = p_{u'}(u_-)$, and $u'''(0) = \xi$. If T_n is bounded, then $u(t)$ is a simple closed characteristic corresponding to the point $(u_-, \bar{u})$, which is a contradiction. Suppose T_n is unbounded. Then the time of either the increasing lap or the decreasing lap is unbounded, and we assume the former without loss of generality. Then u is increasing on $[0, \infty)$ with $\lim_{t\to\infty} u(t) = \alpha$ for some $\alpha \in [u_-, \bar{u}]$ so that $\omega((u_-, 0, p_{u'}(u_-), \xi)) = \{(\alpha, 0, 0, 0)\}$, by Lemma 2.5. This implies that $(\alpha, 0, 0, 0)$ is an equilibrium point, and thus $\alpha = u_*$. Hence there exist times s_1, s_2 such that u satisfies the hypotheses of Theorem 3.3 on $[s_1, s_2]$ which implies that u oscillates around u_* on $[s_1, s_2]$, contradicting the monotonicity of u. □

Lemma 2.7. *There exists $\epsilon > 0$, independent of $\lambda \in [0, 1]$, a set $S_\epsilon = \{(u_1, u_2) \mid |u_1 - u_*| < \epsilon \text{ or } |u_2 - u_*| < \epsilon\}$ such that $\pi(\mathcal{L}^\lambda + \cap \mathcal{L}_-^\lambda) \cap S_\epsilon = \emptyset$.*

Proof. Suppose u is a simple closed characteristic of type (u_1, u_*), and let T be the time such that $u(T) = u_*$. Then $u'(T) = 0$, $u''(T) = 0$, and u has a maximum at T. This implies that $u'''(0) = 0$, which contradicts the uniqueness of solutions to the initial value problem since the constant function $u = u_*$ is a solution. Therefore there does not exist a simple closed characteristic of type (u_1, u_*) and likewise of type (u_*, u_2).

Suppose there exist sequences $\lambda_n \to \lambda_0$ and u_n simple closed characteristics of type $(u_1^n, u_2^n) \in B$ with periods T_n such that $(u_1^n, u_2^n) \to (\bar{u}, u_*)$ as $n \to \infty$. By Lemma 2.4, $\bar{u} \neq u_*$. Moreover, $u_n(0) = u_1^n$, $u_n'(0) = 0$, $u_n''(0) = p_{u'}(u_1^n)$, and $u_n'''(0) = \xi_n \to \xi$ by Proposition 2.3. Let $u(t)$ be the solution with $\lambda = \lambda_0$ and $u(0) = \bar{u}$, $u'(0) = 0$, $u''(0) = p_{u'}(\bar{u}) = 0$, and $u'''(0) = \xi$. If

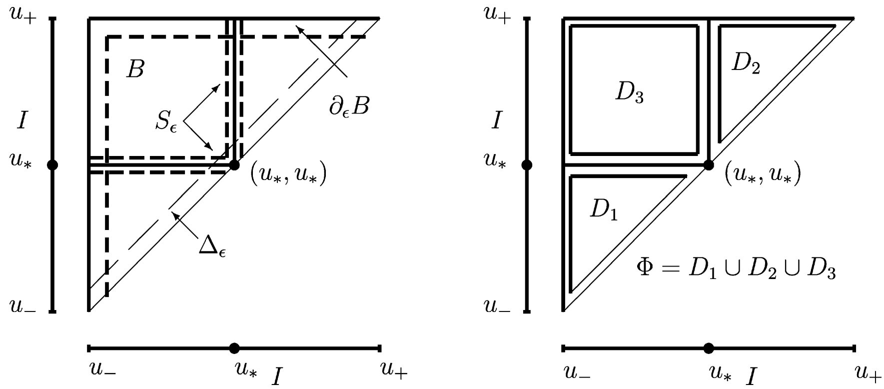

Fig. 2. Simple closed characteristics must have extrema in $\Phi = D_1 \cup D_2 \cup D_3$ as shown in Lemmas 2.4, 2.6, 2.7.

T_n is bounded, then u is a simple closed characteristic of type $(\bar{u}, u_*)$, which is a contradiction. Suppose T_n is unbounded, and assume without loss of generality that the increasing lap time is unbounded. Then u is increasing on $[0, \infty)$, and $\omega((\bar{u}, 0, 0, \xi)) = \{(\alpha, 0, 0, 0)\}$ by Lemma 2.5. This implies that $(\alpha, 0, 0, 0)$ is an equilibrium point, and thus $\alpha = u_*$. Thus there exist times $s_1, s_2 > 0$ satisfying the hypothesis of Theorem 3.3 which implies that u oscillates around u_* on $[s_1, s_2]$ which contradicts the monotonicity of u. $\square$

Theorem 2.8. *Suppose K_λ satisfies hypotheses* (H1)–(H3) *for each $\lambda = [0, 1]$ with $\gamma, C(|u|)$ as well as $K_\lambda(u, 0)$ independent of λ. Let $I = [u_-, u_+]$ be an interval component of πN_E such that I contains exactly one equilibrium $u_* \in (u_-, u_+)$ which is a saddle-focus for each $\lambda \in [0, 1]$. Let $\Phi \subset \operatorname{int}(B)$ be the compact set $B \setminus (\Delta_\epsilon \cup \partial_\epsilon B \cup S_\epsilon)$, where Δ_ϵ, $\partial_\epsilon B$, and S_ϵ satisfy Lemmas* 2.4, 2.6, *and* 2.7, *as shown in Fig.* 2. *Then $\pi(\mathcal{L}^\lambda{}_+ \cap \mathcal{L}^\lambda{}_-) \subset \Phi$ for all $\lambda \in [0, 1]$.*

Proof. The result follows from Lemmas 2.4, 2.6, and 2.7. $\square$

2.3. Twist systems

We now discuss a special class of Lagrangian systems for which the existence of simple closed characteristics has been proved in [19]. Consider a compact interval component I, which possibly contains the projection of a critical point in its interior. Let $\Delta = \{(u_1, u_2) \in I \times I \mid u_1 = u_2\}$, the diagonal of $I \times I$. A second-order Lagrangian system satisfying (H1) is a *twist system* as in [19] if

(T) $\inf\{J_E[u] = \int_0^\tau [\frac{1}{2}|u''|^2 + K(u(t), u'(t)) + E]\,dt \mid \tau \in \mathbb{R}^+,\ u \in C^2([0, \tau]),\ u(0) = u_1,$ $u(\tau) = u_2,\ u'(0) = u'(\tau) = 0$, and $u'(t) \neq 0$ on $(0, \tau)\}$ has a (unique) minimizer $u(t; u_1, u_2)$ for all $(u_1, u_2) \in I \times I \setminus \Delta$ and u and τ are C^1 functions of (u_1, u_2).

For twist systems one can define $S : \mathbb{R}^2 \to \mathbb{R}$ to be the minimum in (T), i.e. $S(u_1, u_2) = J_E[u(t; u_1, u_2)]$. Then simple closed characteristics, being periodic and critical points of J, correspond to critical points of the function $W : \mathbb{R}^2 \to \mathbb{R}$ defined by $W(u_1, u_2) = S(u_1, u_2) + S(u_2, u_1)$, see [19]. Lemma 15 in [19] implies existence of simple closed characteristics for twist systems in the singular case, W has at least one maximum in each of the regions Ω_1 and Ω_2, and W has a saddle point in the region Ω_3 where the sets Ω_i are shown in Fig. 3.

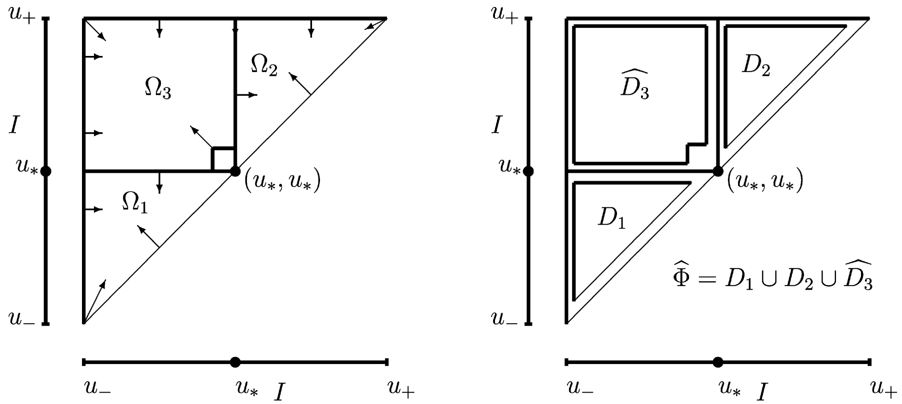

Fig. 3. Twist systems: ∇W on B. The critical points of ∇W are contained in $\widehat{\Phi}$.

The degree of ∇W is easily obtained from Fig. 3 which shows the direction of ∇W on the boundaries of the domains Ω_i as proved in [19]. Using a simple Conley index computation [16], $\deg(\nabla W, \Omega_i, 0) = \pm 1$ for each $i = 1, 2, 3$. By Theorem 2.8, critical points of ∇W are contained in $\widehat{\Phi} = D_1 \cup D_2 \cup \widehat{D_3}$ as shown in Fig. 3. Note that since no critical points of ∇W lie in $D_3 \setminus \widehat{D_3}$, we have $\deg(\nabla W, D_3, 0) = \deg(\nabla W, \widehat{D_3}, 0) = \deg(\nabla W, \Omega_3, 0) = \pm 1$ and $\deg(\nabla W, D_i, 0) = \deg(\nabla W, \Omega_i, 0) = \pm 1$ for $i = 1, 2$, where the regions D_i are shown in Fig. 2.

Lemma 2.9. *Let L be a C^2 Lagrangian satisfying* (H1) *and the twist condition* (T). *Suppose $D \subset \operatorname{int}(B)$ and is compact. Then* $\deg(F, E, 0) = -\deg(\nabla W, D, 0)$ *where F is the map defined in Section* 2.4 *and E is a bounded subset of* $\operatorname{int}(P_+ \times P_-)$ *containing* $F^{-1}(0,0,0,0)$ *with* $\pi(E) = D$.

Proof. Assume that the critical points of F are nondegenerate. Let $F_{ij} : \mathbb{R}^4 \to \mathbb{R}^2$ be the projection of F onto the ith and jth components, so that

$$F_{13}(u_1, p_{u_1}, u_2, p_{u_2}) = \big[u_2 - f_+(u_1, p_{u_1}), u_1 - f_-(u_2, p_{u_2})\big]$$

and

$$F_{24}(u_1, p_{u_1}, u_2, p_{u_2}) = \big[p_{u_2} - g_+(u_1, p_{u_1}), p_{u_1} - g_-(u_2, p_{u_2})\big].$$

Since the twist condition is satisfied,

$$\frac{\partial F_{13}}{\partial(p_{u_1}, p_{u_2})} = \begin{bmatrix} -\frac{\partial f_+}{\partial p_{u_1}} & 0 \\ 0 & -\frac{\partial f_-}{\partial p_{u_2}} \end{bmatrix}$$

is invertible so that locally

$$\begin{bmatrix} p_{u_1} \\ p_{u_2} \end{bmatrix} = P(u_1, u_2) \quad \text{with } DP = -\left(\frac{\partial F_{13}}{\partial(p_{u_1}, p_{u_2})}\right)^{-1} \frac{\partial F_{13}}{\partial(u_1, u_2)}$$

by the Implicit Function Theorem. Then Lemma 5 of [19] implies that

$$\nabla W(u_1,u_2)=\left[-p_{u_1}^+ + p_{u_1}^-, p_{u_2}^+ - p_{u_2}^-\right]=\begin{bmatrix}0 & -1\\ -1 & 0\end{bmatrix} F_{24}\bigl(u_1, P_1(u_1,u_2), u_2, P_2(u_1,u_2)\bigr).$$

Therefore,

$$\begin{bmatrix}0 & -1\\ -1 & 0\end{bmatrix} D\nabla W=\left(\frac{\partial F_{24}}{\partial(p_{u_1},p_{u_2})}\right)DP+\frac{\partial F_{24}}{\partial(u_1,u_2)}.$$

Now we want to relate $\det(DF)$ to $\det(D\nabla W)$. Let Q be the 4×4 permutation which swaps the second and third components and define $\widehat{F}=Q\circ F\circ Q$. Then $\det(DF)=\det(D\widehat{F})$, and

$$D\hat{F}=\begin{bmatrix} -\frac{\partial f_+}{\partial u_1} & 1 & -\frac{\partial f_+}{\partial p_{u_1}} & 0\\ 1 & -\frac{\partial f_-}{\partial u_2} & 0 & -\frac{\partial f_-}{\partial p_{u_2}}\\ -\frac{\partial g_+}{\partial u_1} & 0 & -\frac{\partial g_+}{\partial p_{u_1}} & 1\\ 0 & -\frac{\partial g_-}{\partial u_2} & 1 & -\frac{\partial g_-}{\partial p_{u_2}} \end{bmatrix}=\left[\begin{array}{c|c} A & B\\ \hline C & G\end{array}\right].$$

From the above computations $\det(D\nabla W)=-\det(G(-B^{-1}A)+C)$. Directly computing the 4×4 determinant yields

$$\det(D\widehat{F})=\det\bigl(G\bigl(-B^{-1}A\bigr)B+CB\bigr)=\det\bigl(G\bigl(-B^{-1}A\bigr)+C\bigr)\det(B).$$

Note that there is no general formula for a 4×4 determinant in terms of determinants of its 2×2 blocks, but the special form of $D\widehat{F}$ gives the above formula. Therefore

$$\det(DF)=-\det(D\nabla W)\det\left(\frac{\partial F_{13}}{\partial(p_{u_1},p_{u_2})}\right).$$

In the twist case, $\det(\partial F_{13}/\partial(p_{u_1},p_{u_2}))>0$, so that $\operatorname{sgn}(\det(DF))=-\operatorname{sgn}(\det(D\nabla W))$. Therefore, the critical points of ∇W and F correspond and nondegeneracy with respect to F implies nondegeneracy with respect to ∇W. Therefore, in the nondegenerate case, $\deg(F,E,0)=-\deg(\nabla W,D,0)$, which by standard degree theory implies that this relationship holds in general. □

Thus, in the twist case, on a compact interval component with one saddle-focus equilibrium we obtain at least three closed characteristics, one corresponding to a saddle-point and two corresponding to maximum points of W. Moreover, each of critical points is contained in a neighborhood D such that $\deg(\nabla W,D,0)=\pm 1$, and hence $\deg(F,E,0)=\mp 1$.

The proofs of the previous lemmas can be immediately extended to the case of more than one saddle-focus equilibrium. Thus, combining these results with the fact that in a twist system the number of simple closed characteristics over a compact interval component with e saddle-foci is bounded below by $2e+1$, as shown in [19], yields the following result.

Theorem 2.10. *Suppose L is a C^2 Lagrangian satisfying* (H1) *and* (T). *Let I be a compact interval with e saddle-focus equilibria. Then W has at least $e+1$ maximum values and e saddle points* *which are contained in neighborhoods $D \subset \mathrm{int}(B)$ such that* $\deg(\nabla W, D, 0) = \pm 1$. *Moreover,* $\deg(F, E, 0) = \mp 1$ *where F is defined in Section* 2.4 *and E is a bounded subset of* $\mathrm{int}(P_+ \times P_-)$ *containing* $F^{-1}(0,0,0,0)$ *with* $\pi(E) = D$.

2.4. Existence of simple closed characteristics in the nontwist case

We are now ready to prove the existence of closed characteristics in the nontwist case, and we continue in the same setting as the previous section. Let $F_\lambda : P_+^\lambda \times P_-^\lambda \to \mathbb{R}^4$ be the broken geodesic map for the Lagrangian L_λ with $\lambda \in [0,1]$.

Lemma 2.11. *The level set* $F_\lambda^{-1}(0,0,0,0) = \mathcal{L}_+^\lambda \cap \mathcal{L}_-^\lambda$ *is bounded independently of* $\lambda \in [0,1]$ *and contained in* $\mathrm{int}(P_+^\lambda \times P_-^\lambda)$.

Proof. Let u be a simple closed characteristic for some $\lambda \in [0,1]$ with critical points at $t = 0$ and $t = s$. In Proposition 2.3 it is shown that $u'''(0)$ and $u'''(s)$ are bounded independently of λ, which implies that $p_{u_1}^+ = \partial_{u'} K_\lambda(u_1, 0) - u'''(0)$ and $p_{u_2}^- = \partial_{u'} K_\lambda(u_2, 0) - u'''(s)$ are bounded independently of λ. Thus $F_\lambda^{-1}(0,0,0,0)$ is bounded in $P_+^\lambda \times P_-^\lambda$. Suppose $(u_1, p_{u_1}, u_2, p_{u_2}) \in \mathcal{L}_+^\lambda \cap \mathcal{L}_-^\lambda$. Let u be the corresponding simple closed characteristic of type (u_1, u_2) with $u(0) = u_1$, $u'(0) = 0$, $u''(0) = p_{u'}(u_1)$, and $u'''(0) = \partial_{u'} K_\lambda(u_1, 0) - p_{u_1}$. Then there exists a time T such that $u'(T) < 0$, and $\max u(t) = u_2 \in \mathrm{int}(I)$ by Theorem 2.8. If $\hat{u}$ is any nearby solution to (2) with $\hat{u}(0) = \hat{u}_1$, $\hat{u}'(0) = 0$, $\hat{u}''(0) = p_{u'}(\hat{u}_1)$ and $\hat{u}'''(0) = \partial_{u'} K_\lambda(\hat{u}_1, 0) - \hat{p}_{u_1}$, then for $(\hat{u}_1, \hat{p}_{u_1})$ close enough to (u_1, p_{u_1}), we have $\hat{u}'(T) < 0$ and $\max_{t \in [0,T]} \hat{u}(t) \in \mathrm{int}(I)$, so that $(\hat{u}_1, \hat{p}_{u_1}) \in P_+^\lambda$. A similar argument holds for (u_2, p_{u_2}), and hence $(u_1, p_{u_1}, u_2, p_{u_2}) \in \mathrm{int}(P_+^\lambda \times P_-^\lambda)$. $\square$

Theorem 2.12. *Let M_E be a singular energy level set of a second-order Lagrangian system with C^2 Lagrangian L satisfying hypotheses* (H1)–(H3). *Suppose there exists a compact interval component I containing a single saddle-focus equilibrium* $u_* \in \mathrm{int}(I)$. *Then there exist at least three simple closed characteristics on M_E.*

Proof. Let $L_0 = L = \frac{1}{2}u''^2 + K(u, u')$ and L_1 be the Swift–Hohenberg Lagrangian defined by $L_1 = \frac{1}{2}u''^2 + K(u, 0)$. We consider a homotopy between these two Lagrangians by $L_\lambda = (1 - \lambda)L_0 + \lambda L_1$ for all $\lambda \in [0,1]$. From the definition of L_λ the level sets $N_\lambda = N$ and the set $B_\lambda = B$ for all $\lambda \in [0,1]$. Since L satisfies hypotheses (H2)–(H3), it is straightforward to check that L_λ satisfies (H2)–(H3) for all $\lambda \in [0,1]$, with γ, $C(|u|)$ independent of λ.

Since L_1 is a twist system, Lemma 15 of [19] implies that there exists a simple closed characteristic in each of the domains D_1, D_2, and D_3 shown in Fig. 2. In particular, Lemma 2.11 and Theorem 2.8 imply that for each D_i with $i = 1,2,3$, there exists a compact domain $E_i \subset \mathrm{int}(P_+^1 \times P_-^1)$ with $\pi(E_i) = D_i$ such that $\mathcal{L}_+^1 \cap \mathcal{L}_-^1 \subset E_1 \cup E_2 \cup E_3$. Then Lemma 2.9 implies $\deg(F_1, E_i, 0) \neq 0$.

Furthermore, Theorem 2.8 implies that the domains D_1, D_2, and D_3 can be chosen independently of λ so that $\pi(\mathcal{L}_+^\lambda \cap \mathcal{L}_-^\lambda) \subset D_1 \cup D_2 \cup D_3$ for all $\lambda \in [0,1]$. Lemma 2.11 implies that for each $i = 1,2,3$ the compact domain E_i can be chosen so that $E_i \subset \mathrm{int}(P_+^\lambda \times P_-^\lambda)$ for all $\lambda \in [0,1]$ with $\pi(E_i) = D_i$ such that $F_\lambda : E_i \to \mathbb{R}^4$ is continuous and $\mathcal{L}_+^\lambda \cap \mathcal{L}_-^\lambda \subset E_1 \cup E_2 \cup E_3$ for all $\lambda \in [0,1]$. The homotopy invariance property of the degree now implies that for each $\lambda \in [0,1]$ we have $\deg(F_\lambda, E_i, 0) = \deg(F_1, E_i, 0) \neq 0$. This implies that for the Lagrangian $L_0 = L$ there exists a simple closed characteristic in each D_i for $i = 1,2,3$. $\square$

Theorem 2.12 can be extended to prove the existence of simple closed characteristics for singular energy levels with multiple singular points corresponding to saddle-focus equilibria, as stated in Theorem 1.2. The proof is identical using Theorem 2.10 for twist systems. Therefore, the remaining work is to prove Theorem 2.2.

3. Saddle-focus equilibria

In this section, we describe some local results about solutions near saddle-focus equilibria which are needed in Section 4. Less general versions of these results were first proved in [10], but here Theorem 3.2 incorporates estimates from [4], and Theorem 3.3 extends Theorem 4.2 in [10] to include estimates on derivatives.

Suppose u_* is an equilibrium point of Eq. (2) so that $\partial_u K(u_*, 0) = 0$ and $K(u_*, 0) + E = 0$. Then expanding $K(u, u') + E$ around the point $(u_*, 0)$ gives

$$\begin{aligned} K(u,u') + E = {} & K(u_*,0) + E + \partial_u K(u_*,0)(u-u_*) + \partial_{u'} K(u_*,0)u' \\ & + \frac{1}{2}\partial_u^2 K(u_*,0)(u-u_*)^2 + \partial_{uu'}^2 K(u_*,0)(u-u_*)u' \\ & + \frac{1}{2}\partial_{u'}^2 K(u_*,0)(u')^2 + R(u,u'). \end{aligned}$$

Integrating over a finite interval $[u_1, u_2]$, the terms $\partial_{u'} K(u_*, 0)u'$ and $\partial_{uu'}^2 K(u_*, 0)(u - u_*)u'$ integrate to constants depending only on u_1 and u_2. Thus minimizing the functional J_E is the same as minimizing $\widehat{J}_E[u] = \int_0^\tau [\frac{1}{2}|u''|^2 + \widehat{K}(u, u') + E]\,dt$ where

$$\widehat{K}(u,u') + E = \frac{1}{2}\partial_{u'}^2 K(u_*,0)(u')^2 + \frac{1}{2}\partial_u^2 K(u_*,0)(u-u_*)^2 + R(u,u').$$

Therefore near equilibrium points we will assume that J is of the form

$$J[u] = \int_0^T \left[\frac{1}{2}|u''|^2 + \frac{1}{2}\beta|u'|^2 + \frac{1}{2}\alpha|u-u_*|^2 + R(u,u')\right]dt \tag{3}$$

for some constants $\alpha > 0$ and $\beta \in \mathbb{R}$ where the remainder $R(u, u')$ consists of terms which are at least cubic in $(u - u_*, u')$.

For convenience, in this section we take the equilibrium point to be the origin. Then we can choose $0 < \delta_1 < 1$ such that $R(u, v) = u^2 g(u, v) + v^2 h(u, v)$ for $\|(u, v)\|_\infty < \delta_1$ with $|g(u, v)| \leqslant \frac{1}{4}\alpha$ and $|h(u, v)| \leqslant \frac{1}{4}|\beta|$ when $\beta \neq 0$. In the case $\beta = 0$, we choose $\delta_1 > 0$ so that $|h(u, v)| \leqslant \frac{1}{2}\sqrt{\alpha}$. We then consider the action $J[u]$ in Eq. (3) with $u_* = 0$ for $u \in X_{T,\delta_1}(x, y) = \{u \in H^2([0, T]) \mid (u(0), u'(0)) = x,\ (u(T), u'(T)) = y,\ \text{and } \|(u, u')\|_\infty \leqslant \delta_1\}$. The first lemma is due to Bonheure [4] and included for completeness.

Lemma 3.1. *(See [4].) Let $\alpha > 0$ and $\beta \in (-\sqrt{4\alpha}, 0]$. Then there exists $\epsilon(\alpha, \beta) > 0$ such that for any $u \in H^2(a, b)$, we have*

$$\int_a^b \left[\frac{1}{2}|u''|^2 + \frac{1}{2}\beta|u'|^2 + \frac{1}{2}\alpha u^2\right] dt \geqslant \epsilon \|u\|^2_{H^2(a,b)} - \left(\epsilon - \frac{1}{2}\beta\right)[uu']_a^b.$$

Proof. For any constant $k \in \mathbb{R}$

$$\int_a^b (u'' + ku)^2\, dt = \int_a^b \left[|u''|^2 - 2k|u'|^2 + k^2u^2\right] dt + 2k[u'u]_a^b.$$

We then estimate

$$\begin{aligned}
&\int_a^b \left[\frac{1}{2}|u''|^2 + \frac{1}{2}\beta|u'|^2 + \frac{1}{2}\alpha u^2\right] dt \\
&\quad = \epsilon \int_a^b \left[|u''|^2 + |u'|^2 + u^2\right] dt \\
&\qquad + \frac{1-2\epsilon}{2} \int_a^b \left[|u''|^2 - \left(\frac{2\epsilon - \beta}{1 - 2\epsilon}\right)|u'|^2 + \frac{1}{4}\left(\frac{2\epsilon - \beta}{1 - 2\epsilon}\right)^2 u^2\right] dt \\
&\qquad + \left[\frac{\alpha}{2} - \epsilon - \frac{(2\epsilon - \beta)^2}{8(1 - 2\epsilon)}\right] \int_a^b u^2\, dt.
\end{aligned}$$

For $\epsilon < \frac{1}{2}$ small enough, we have $\frac{\alpha}{2} - \epsilon - \frac{(2\epsilon-\beta)^2}{8(1-2\epsilon)} \geqslant 0$, and choosing $2k = \frac{2\epsilon-\beta}{1-2\epsilon}$ yields the desired estimate. □

In the next two theorems we establish the existence of minimizers of J near a saddle-focus equilibrium. Moreover, these minimizers, and indeed any solutions near a saddle-focus, must oscillate within a fixed time.

Theorem 3.2. *Suppose J is in the form of* (3) *with $\alpha > 0$ and $\beta^2 < 4\alpha$. Then there exists $\delta_0 > 0$ such that if $\|x\|, \|y\| \leqslant \delta < \delta_0$ and $T \geqslant 1$, then there exists a unique global minimizer $\hat{u}$ of J in $X_{T,\delta_1}(x, y)$. Furthermore, $\|\hat{u}\|_{W^{3,\infty}} \leqslant C\delta$ and $J[\hat{u}] \leqslant C\delta^2$, where $C > 0$ is independent of $T \geqslant 1$.*

Proof. We divide the proof into several estimates.

Step 1: There exists $C_1(\alpha, \beta, \delta_1) > 0$ such that if $\|x\|, \|y\| \leqslant \delta < \delta_1$, then $\inf_{X_{T,\delta_1}(x,y)} J \leqslant C_1\delta^2$.

Choose any functions $\rho_0, \rho_1 \in C^\infty([0,T])$ such that $\operatorname{supp}(\rho_j) \subset [0,1/2]$ with $\rho_0(0)=1$, $\rho_0'(0)=0$, $\rho_1(0)=0$, and $\rho_1'(0)=1$, and define $\psi_j(t)=(-1)^j\rho_j(T-t)$. Consider the function $\phi \in X_{T,\delta_1}(x,y)$ defined by $\phi = x_0\rho_0 + x_1\rho_1 + y_0\psi_0 + y_1\psi_1$. Since $\|(\phi,\phi')\|_\infty < \delta_1$, we have

$$J[\phi] \leqslant \int_0^T \left[\frac{1}{2}|\phi''|^2 + \gamma|\phi'|^2 + \frac{3}{4}\alpha\phi^2\right]dt \quad \text{where } \gamma = \begin{cases} \frac{3}{4}|\beta| & \text{if } \beta \neq 0, \\ \frac{1}{2}\sqrt{\alpha} & \text{if } \beta = 0, \end{cases}$$

and hence $\inf_{X_{T,\delta_1}(x,y)} J \leqslant J[\phi] \leqslant C_1\delta^2$. Note that C_1 is independent of T.

Step 2: There exist $\delta_0 < \delta_1/2$ and $C(\delta_0)$ such that, if $\|x\|, \|y\| \leqslant \delta < \delta_0$ and $u \in X_{T,\delta_1}(x,y)$ with $J[u] \leqslant 2\inf_{X_{T,\delta_1}(x,y)} J$, then $\|u\|_\infty, \|u\|_{H^2} \leqslant C\delta$.

If $\beta > 0$, the functional $J^{1/2}[u]$ is equivalent to $\|u\|_{H^2}$ on $X_{T,\delta_1}(x,y)$. Indeed

$$\begin{aligned} J[u] &= \int_0^T \left[\frac{1}{2}|u''|^2 + \left(\frac{1}{2}\beta + h(u,u')\right)|u'|^2 + \left(\frac{1}{2}\alpha + g(u,u')\right)u^2\right]dt \\ &\geqslant \int_0^T \left[\frac{1}{2}|u''|^2 + \frac{1}{4}\beta|u'|^2 + \frac{1}{4}\alpha u^2\right]dt \geqslant C(\alpha,\beta)\|u\|_{H^2}^2 \end{aligned}$$

which yields $\|u\|_{H^2} \leqslant C\delta$.

In the case $\beta < 0$, we let $I_2[u] = \int_0^T [\frac{1}{2}|u''|^2 + \frac{3}{4}\beta|u'|^2 + \frac{1}{4}\alpha u^2]\,dt$ so that for $\|(u,u')\|_\infty < \delta_1$ we have $J[u] \geqslant I_2[u]$. By Lemma 3.1 we have

$$\epsilon\|u\|_{H^2[0,T]}^2 \leqslant I_2[u] + \left|\left(\epsilon - \frac{1}{2}\beta\right)[uu']_0^T\right| \leqslant J[u] + C\delta^2 \leqslant C\delta^2$$

which implies that $\|u\|_{H^2} \leqslant C\delta$.

Finally, in the case $\beta = 0$, we let $I_2[u] = \int_0^T [\frac{1}{2}|u''|^2 - \frac{1}{2}\sqrt{\alpha}|u'|^2 + \frac{1}{4}\alpha u^2]\,dt$ so that for $\|(u,u')\|_\infty < \delta_1$ we have $J[u] \geqslant I_2[u]$. By Lemma 3.1 we have

$$\epsilon\|u\|_{H^2[0,T]}^2 \leqslant I_2[u] + \left|\left(\epsilon - \frac{1}{2}\sqrt{\alpha}\right)[uu']_0^T\right| \leqslant J[u] + C\delta^2 \leqslant C\delta^2$$

which implies that $\|u\|_{H^2} \leqslant C\delta$.

In all cases $\|u\|_{H^2} \leqslant C\delta$ implies that $\|u\|_\infty \leqslant C\delta$ and $\|u'\|_\infty \leqslant C\delta$.

Step 3: For δ_0 sufficiently small, J has a unique minimizer $\hat{u} \in X_{T,\delta_1}(x,y)$ such that $\|\hat{u}\|_{W^{3,\infty}} \leqslant C\delta$ where C is independent of $T \geqslant 1$.

Since $\|(u,u')\|_\infty < \delta_1$ for $u \in X_{T,\delta_1}(x,y)$ with $J[u] \leqslant 2\inf_{X_{T,\delta_1}(x,y)} J$ and $\|x\|, \|y\| \leqslant \delta$, we have that J is weakly lower semicontinuous on $X_{T,\delta_1}(x,y)$ and coercive on $\{u \mid J[u] \leqslant 2\inf_{X_{T,\delta_1}(x,y)} J\}$. Thus a minimizer $\hat{v}$ can be found by standard theory which is a solution to the differential equation. From the differential equation we obtain the estimate

$$\|\hat{v}''''\|_{L^2} \leqslant C\delta$$

and by interpolation we have

$$\|\hat{v}'''\|_{L^2} \leqslant C\big(\|\hat{v}''''\|_{L^2} + \|\hat{v}''\|_{L^2}\big) \leqslant C\delta.$$

Therefore we have a bound on the H^4-norm which implies a bound on the $W^{3,\infty}$-norm. □

Theorem 3.3. *Suppose $\beta^2 < 4\alpha$ so that the origin is a saddle-focus equilibrium in the four-dimensional flow. Then there exist $\delta_0 > 0$ and $\tau_0 > 0$ such that every solution u to the Euler–Lagrange equation corresponding to J in* (3) *with $\|u\|_{W^{3,\infty}([0,T])} < \delta_0$ changes sign in any subinterval of length τ_0 in $[0, T]$. In particular, the unique global minimizer $\hat{v}$ of J in $X_{T,\delta_1}(x, y)$ with $\delta < \delta_0$ satisfying Theorem* 3.2 *changes sign in any subinterval of length τ_0 in $[0, T]$ for $T \geqslant 1$.*

Proof. First we consider solutions to the linear differential equation

$$w'''' - \beta w'' + \alpha w = 0. \tag{4}$$

Since the origin is a saddle-focus, it has complex eigenvalues $\pm\lambda \pm \mu i$. By rescaling time we can assume without loss of generality that $\mu = 1$ and $\lambda > 0$. Therefore all solutions to (4) have the form

$$w(t) = Ae^{-\lambda t}\sin(t + \phi) + Be^{\lambda t}\sin(t + \psi)$$

for some A, B, ϕ, and ψ. Also

$$w'(t) = -\lambda Ae^{-\lambda t}\sin(t + \phi) + Ae^{-\lambda t}\cos(t + \phi) + \lambda Be^{\lambda t}\sin(t + \psi) + Be^{\lambda t}\cos(t + \psi).$$

Step 1: There exists $\tau_0 > 0$ depending only on λ such that for every A, B, ϕ, and ψ there are points $\tau_\pm \in [0, \tau_0]$ such that

$$\pm w(\tau_\pm) \geqslant \frac{1}{\tau_0}\big\|(w, w')\big\|_{L^\infty[0,\tau_\pm]}.$$

We prove only the existence of $\tau = \tau_+$, as the other case is similar. The calculation is separated into two cases. First suppose

$$|B|e^{2\pi\lambda} \leqslant \frac{1}{2}|A|e^{-2\pi\lambda}.$$

Choose $\tau \in [0, 2\pi]$ such that $\sin(\tau + \phi) = \operatorname{sgn} A$. Then we can estimate

$$w(\tau) \geqslant |A|e^{-2\pi\lambda} - |B|e^{2\pi\lambda} \geqslant \frac{1}{2}|A|e^{-2\pi\lambda},$$

$$\|w\|_{L^\infty[0,\tau]} \leqslant |A| + |B|e^{2\pi\lambda} \leqslant |A| + \frac{1}{2}|A|e^{-2\pi\lambda} \leqslant 2|A|,$$

and

$$\|w'\|_{L^\infty[0,\tau]} \leqslant \lambda|A| + |A| + \lambda|B|e^{2\pi\lambda} + |B|e^{2\pi\lambda} \leqslant 2(1 + \lambda)|A|.$$

Otherwise

$$|B|e^{2\pi\lambda} \geqslant \frac{1}{2}|A|e^{-2\pi\lambda}.$$

Choose $\tau \in [2\pi + \lambda^{-1}\ln 4, 4\pi + \lambda^{-1}\ln 4]$ such that $\sin(t+\psi) = \operatorname{sgn} B$. For this choice of τ we have

$$\frac{1}{2}|B|e^{\lambda\tau} \geqslant 2|B|e^{2\pi\lambda} \geqslant |A|e^{-2\pi\lambda} \geqslant |A|e^{-\lambda\tau}.$$

Thus we can estimate

$$w(\tau) \geqslant |B|e^{\lambda\tau} - |A|e^{-\lambda\tau} \geqslant \frac{1}{2}|B|e^{\lambda\tau} \geqslant \frac{1}{2}|B|e^{2\pi\tau},$$
$$\|w\|_{L^\infty[0,\tau]} \leqslant |A| + |B|e^{\lambda\tau} \leqslant 2|B|e^{4\pi\lambda} + |B|e^{\lambda\tau} \leqslant |B|\big[2e^{4\pi\lambda} + e^{4\pi+\lambda^{-1}\ln 4}\big],$$

and

$$\begin{aligned}\|w'\|_{L^\infty[0,\tau]} &\leqslant \lambda|A| + |A| + \lambda|B|e^{\lambda\tau} + |B|e^{\lambda\tau} \leqslant 2(1+\lambda)|B|e^{4\pi\lambda} + |B|e^{\lambda\tau}\\ &\leqslant (1+\lambda)|B|\big[2e^{4\pi\lambda} + e^{4\pi+\lambda^{-1}\ln 4}\big].\end{aligned}$$

If τ_0 is chosen larger than

$$\max\left\{4\pi + \frac{\ln 4}{\lambda}, 4e^{2\pi\lambda} + 2e^{4\pi-2\pi\lambda+\lambda^{-1}\ln 4}, (1+\lambda)4e^{2\pi\lambda} + 2e^{4\pi-2\pi\lambda+\lambda^{-1}\ln 4}\right\} \geqslant 1,$$

then for every w there is $\tau_+ \in [0, \tau_0]$ such that

$$w(\tau_+) \geqslant \frac{1}{\tau_0}\big\|(w, w')\big\|_{L^\infty([0,\tau_+])}.$$

Step 2: There exists $\delta_2 > 0$ such that if v is the solution to the nonlinear differential equation

$$v'''' - \beta v'' + \alpha v + \partial_v R(v, v') - \partial^2_{vv'} R(v, v')v' - \partial^2_{v'} R(v, v')v'' = 0$$

with initial conditions $v_0 = (v(0), v'(0), v''(0), v'''(0))$ and $\|v\|_{W^{3,\infty}([0,\tau_0])} < \delta_2$, then v changes sign in $[0, \tau_0]$.

By the variation of constants formula we have

$$\mathbf{v}(t) = \mathbf{w}(t) + \int_0^t e^{\mathbf{L}(t-s)}\mathbf{N}\big(v(s)\big)\,ds,$$

$$\mathbf{v}'(t) = \mathbf{w}'(t) + \mathbf{N}\big(v(t)\big) + \int_0^t \mathbf{L}e^{\mathbf{L}(t-s)}\mathbf{N}\big(v(s)\big)\,ds,$$

where $\mathbf{v} = (v, v', v'', v''')$, $\mathbf{w} = (w, w', w'', w''')$, $\mathbf{N}(v) = (0, 0, 0, -\partial_v R(v, v') + \partial^2_{vv'} R(v, v')v' + \partial^2_{v'} R(v, v')v'')$, and $\mathbf{L}$ is the linear part of the vector field. Since $|\partial_v R| = O(\|(v, v')\|^2)$, $|\partial^2_{vv'} Rv'| = O(\|(v, v')\|^2)$, and $|\partial^2_{v'} Rv''| = O(\|(v, v')\| \cdot |v''|)$, for all $K \leqslant 1$ there exists $\delta_2 > 0$ such that

$$\|\mathbf{N}(v(s))\| \leqslant K\|(v, v')\|_\infty$$

if $\|v\|_{W^{3,\infty}([0,\tau_0])} < \delta_2$. Let

$$C = \max\{\tau_0\|e^{L\tau_0}\|, 1 + \tau_0\|Le^{L\tau_0}\|\}.$$

Then for $t \in [0, \tau_0]$ we have

$$\|v - w\|_{L^\infty([0,t])} \leqslant C \cdot K(\delta_2)\|(v, v')\|_{L^\infty([0,t])}$$

and

$$\|v' - w'\|_{L^\infty([0,t])} \leqslant C \cdot K(\delta_2)\|(v, v')\|_{L^\infty([0,t])}.$$

Now choose $\delta_2 > 0$ such that for $K = K(\delta_2)$ we have $0 < C \cdot K/(1 - C \cdot K) \leqslant 1/2\tau_0$. We now estimate as follows,

$$\begin{aligned}\|(v, v')\|_{L^\infty([0,t])} &\leqslant \|(w, w)\|_{L^\infty([0,t])} + \|(v, v') - (w, w')\|_{L^\infty([0,t])} \\ &\leqslant \|(w, w')\|_{L^\infty([0,t])} + C \cdot K\|(v, v')\|_{L^\infty([0,t])},\end{aligned}$$

and hence

$$(1 - C \cdot K)\|(v, v')\|_{L^\infty([0,t])} \leqslant \|(w, w')\|_{L^\infty([0,t])}.$$

This implies that

$$\begin{aligned}\|(v, v') - (w, w')\|_{L^\infty([0,t])} \leqslant C \cdot K\|(v, v')\|_{L^\infty([0,t])} &\leqslant \frac{C \cdot K}{1 - C \cdot K}\|(w, w')\|_{L^\infty([0,t])} \\ &\leqslant \frac{1}{2\tau_0}\|(w, w')\|_{L^\infty([0,t])}.\end{aligned}$$

Now take $t = \tau = \tau_+$ as in Step 1. Then

$$\begin{aligned}v(\tau) &\geqslant w(\tau) - \|(v, v') - (w, w')\|_{L^\infty([0,\tau])} \\ &\geqslant \frac{1}{\tau_0}\|(w, w')\|_{L^\infty([0,\tau])} - \frac{1}{2\tau_0}\|(w, w')\|_{L^\infty([0,\tau])} > 0.\end{aligned}$$

So $v(\tau_+) > 0$, and similarly $v(\tau_-) < 0$.

Finally let $T \geqslant 1$ and $\hat{v}$ be the minimizer from Theorem 3.2 on the interval $[0, T]$. Note in the above analysis we rescaled time, and hence redefine the constant τ_0 to be τ_0/μ. Then either $T < \tau_0$ and the theorem is vacuously satisfied, or $T \geqslant \tau_0$. In the latter case, Step 2 above implies that $\hat{v}$ changes sign on every subinterval of length τ_0 in $[0, T]$. This completes the proof. □

4. Existence of laps across an equilibrium

In this section we establish the existence of laps via minimization to prove Theorem 2.2. We alter minimizing sequences to obtain convergence, extending the techniques in [10] and [11]. One of the tools in this process is clipping, as described in the following lemma. For a proof the reader is referred to [10].

Lemma 4.1. *(See Theorem 3.1 in [10].) Let $a_1 < b_1 \leqslant a_2 < b_2$ and $I_j = [a_j, b_j]$ with $j = 1$ or 2. Suppose a function $u \in C^1(I_1) \cap C^1(I_2)$ is increasing on both I_1 and I_2 with $u(I_1) \cap u(I_2) \neq \emptyset$ and satisfies one of the following two properties*:

(i) $u(a_1) = u(a_2)$, $u(b_1) = u(b_2)$ *and* $(u'(a_1) - u'(a_2)) \cdot (u'(b_1) - u'(b_2)) \leqslant 0$, *or*
(ii) $u'(a_1) = u'(a_2) = u'(b_1) = u'(b_2) = 0$ *and* $(u(a_1) - u(a_2)) \cdot (u(b_1) - u(b_2)) \geqslant 0$.

Then there exists $c_j \in I_j$ such that $u(c_1) = u(c_2)$ and $u'(c_1) = u'(c_2)$.

If $u \in H^2([a_1, b_2])$ satisfies one of the above hypotheses, then the interval $[c_1, c_2]$ can be removed from the domain of u and the two pieces glued at c_1 and c_2 to obtain $\hat{u} \in H^2([a_1, b_2 - (c_2 - c_1)])$.

4.1. The minimization problem

Let $\mathbf{u} = (u_1, u_2)$. Define

$$X_\tau(\mathbf{u}) = \big\{u \in H^2\big([0, \tau]\big) \bigm| u(0) = u_1,\ u(\tau) = u_2,\ u'(0) = 0,\ u'(\tau) = 0,\ \text{and}\ u'(t) \neq 0 \text{ for } t \in (0, \tau)\big\}$$

and the functional

$$J_E[u] = \int_0^\tau \left[\frac{1}{2}|u''|^2 + K\big(u(t), u'(t)\big) + E\right] dt$$

on $X(\mathbf{u}) = \bigcup_{\tau \in \mathbb{R}^+} X_\tau(\mathbf{u})$. Now consider the minimization problem

$$\mathcal{J}_E(\mathbf{u}) = \inf_{\substack{u \in X_\tau \\ \tau \in \mathbb{R}^+}} J_E[u]. \tag{5}$$

In [11] it was shown that if E is a regular value of the Hamiltonian, then there exist minimizers for (5) in $X(\mathbf{u})$, which we refer to as minimizing laps. As explained in the introduction, our first goal is to show that such minimizing laps exist when E is a critical value of the Hamiltonian and there exists exactly one equilibrium point $(u_*, 0, 0, 0)$ for the Euler–Lagrange equations with $u_* \in (u_1, u_2)$. This result will then be extended to a finite number of such critical points.

In this section, we work with increasing laps with $u_1 < u_2$, but the arguments for decreasing laps are the same. Also, we use the following notation. For $u \in X(\mathbf{u})$ and $\mu \in [u_1, u_2]$ we let $t(\mu)$ denote the unique time at which $u(t(\mu)) = \mu$.

4.2. Bounding time away from singular values

In this section, we establish bounds on the functional J which are required for minimization.

Lemma 4.2. *(See Lemma 3.1 in [11].) If hypothesis* (H3) *holds then for every $\epsilon > 0$ there exists $C_\epsilon \geqslant 0$ such that $K(u,v) + E + \epsilon^{-1} v^4 \geqslant -C_\epsilon |v|$ for all $u \in I$ and $v \in \mathbb{R}$.*

Lemma 4.3. *If $u \in X(\mathbf{u})$ with $\mu \in [u_1, u_2]$, then*

$$\int_0^{t(\mu)} |u''|^2\,dt \geqslant \frac{4}{9|\mu - u_1|^2} \int_0^{t(\mu)} |u'|^4\,dt \quad \textit{and} \quad \int_{t(\mu)}^{\tau} |u''|^2\,dt \geqslant \frac{4}{9|u_2 - \mu|^2} \int_{t(\mu)}^{\tau} |u'|^4\,dt.$$

Proof. Since u is monotone, we can reparametrize by $u'(t) = v(u)$ and let $z(u) = v|v|^{1/2}(u)$. We consider the case where u is increasing. Transforming to (u,z)-variables yields

$$\int_0^{t(\mu)} \left|u''(t)\right|^2 dt = \frac{4}{9} \int_{u_1}^{\mu} \left|z'(u)\right|^2 du \quad \text{and} \quad \int_0^{t(\mu)} \left|u'(t)\right|^4 dt = \int_{u_1}^{\mu} \left|z(u)\right|^2 du$$

so that $z \in H_0^1([u_1, u_2])$. Hence z is absolutely continuous with $z(\mu) - z(u_1) = \int_{u_1}^{\mu} z'(u)\,du$ for all $\mu \in [u_1, u_2]$, which implies $|z(\mu)|^2 \leqslant |\mu - u_1| \int_{u_1}^{\mu} |z'|^2\,du$. Therefore,

$$\int_0^{t(\mu)} |u''|^2\,dt = \frac{4}{9} \int_{u_1}^{\mu} |z'|^2\,du \geqslant \frac{4}{9|\mu - u_1|^2} \int_{u_1}^{\mu} z^2\,du = \frac{4}{9|\mu - u_1|^2} \int_0^{t(\mu)} |u'|^4\,dt.$$

The other inequality is similar. □

Lemma 4.4. *If $\mu \in [u_1, u_2]$, then there exists a constant $C(|u_2 - u_1|) > 0$ such that $J_E[u|_{[0,t(\mu)]}] \geqslant -C$ and $J_E[u|_{[t(\mu),\tau]}] \geqslant -C$ for all $u \in X(\mathbf{u})$.*

Proof. Using Lemma 4.2, we estimate

$$\begin{aligned} J_E[u|_{[0,t(\mu)]}] &= \int_0^{t(\mu)} \left[\frac{1}{2}|u''|^2 + K(u,u') + E\right] dt \geqslant \int_0^{t(\mu)} \left[K(u,u') + E + \frac{2}{9|\mu - u_1|^2}|u'|^4\right] dt \\ &\geqslant -\int_0^{t(\mu)} Cu'\,dt = -C|\mu - u_1| \geqslant -C|u_2 - u_1|. \end{aligned}$$

The other case is similar. □

Define the sublevel set $J_E^a(\mathbf{u}) = \{u \in X(\mathbf{u}) \mid J_E[u] \leqslant a\}$. We have the following lemma from [11], which is included for completeness.

Lemma 4.5. *(See [11].) There exists a constant $C(a, |u_2 - u_1|)$ such that for any $u \in J_E^a(\mathbf{u})$, we have $\|u'\|_{L^4([0,\tau])} \leqslant C$ and $\|u''\|_{L^2([0,\tau])} \leqslant C$.*

Proof. We estimate

$$a \geqslant J_E[u] = \int_0^\tau \left[\frac{1}{2}|u''|^2 + K(u,u') + E\right] dt \geqslant \frac{1}{9|u_2 - u_1|^2} \int_0^\tau |u'|^4\, dt - C|u_2 - u_1|$$

so that $\|u'\|_{L^4}$ is bounded. Now for $\epsilon > 0$, using Lemma 4.2 we estimate

$$a \geqslant J_E[u] = \int_0^\tau \left[\frac{1}{2}|u''|^2 + K(u,u') + E\right] dt \geqslant \frac{1}{2}\|u''\|_{L^2([0,\tau])} - \epsilon^{-1} \int_0^\tau |u'|^4\, dt - C_\epsilon |u_2 - u_1|,$$

which gives the bound on $\|u''\|_{L^2([0,\tau])}$. □

If $\mu_1, \mu_2 \in [u_1, u_2]$ and $[\mu_1, \mu_2]$ contains no critical points of H, we have the following property due to continuity.

(P1) There exist $\rho > 0$ and δ_2 such that $K(u, v) + E \geqslant \rho > 0$ for all $(u, v) \in [\mu_1, \mu_2] \times [-\delta_2, \delta_2]$.

The following lemma is a variation of Lemma 3.9 from [11].

Lemma 4.6. *Suppose that $[\mu_1, \mu_2]$ contains no critical points of H. Under hypotheses* (H1) *and* (H3)*, there exists a constant $T > 0$, depending on $a, |u_2 - u_1|, 1/\delta_2$, and $1/\rho$, such that $|t(\mu_2) - t(\mu_1)| \leqslant T$.*

Proof. Let $S_{\delta_2} = \{t \in [t(\mu_1), t(\mu_2)] \mid |u'(t)| \geqslant \delta_2\}$, where δ_2 is chosen in (P1). Since $|S_{\delta_2}|\delta_2^4 \leqslant \int_{t(\mu_1)}^{t(\mu_2)} |u'|^4\, dt$, Lemma 4.5 implies $|S_{\delta_2}| \leqslant C(a, |u_2 - u_1|, 1/\delta_2^4)$. Let $\epsilon > 0$. Then,

$$\begin{aligned} a \geqslant J_E[u] &\geqslant \int_{S_{\delta_2}^c} \left[K(u,u') + E\right] dt + \int_{S_{\delta_2}} \left[K(u,u') + E\right] dt \\ &\geqslant \rho\left(\left|t(\mu_2) - t(\mu_1)\right| - |S_{\delta_2}|\right) - \epsilon^{-1} \int_{S_{\delta_2}} |u'|^4\, dt - C_\epsilon |\mu_2 - \mu_1| \end{aligned}$$

by Lemma 4.2, which implies $|t(\mu_2) - t(\mu_1)| \leqslant T(a, |u_2 - u_1|, 1/\delta_2^4, 1/\rho)$. □

4.3. Existence of minimizing laps

To show existence of a minimizer we want to find $u \in X(\mathbf{u})$ such that $J_E[u] < \mathcal{J}_E + \epsilon$ with a global bound on time T independent of $\epsilon > 0$ so that we can form a minimizing sequence that has a uniform bound on time. We will do this by choosing an appropriate δ-neighborhood around the equilibrium point and show that we can, if necessary, modify the function u on that neighborhood to produce a function with less action for which the time spent in the δ-neighborhood is a priori bounded. Lemma 4.6 provides an a priori bound on time outside the δ-neighborhood. With this we have the following theorem.

Theorem 4.7. *Suppose $u_* \in (u_1, u_2)$ is a saddle-focus and is the only equilibrium point in $[u_1, u_2]$. Then there exists $T > 0$ such that for every $\epsilon > 0$ there exists a strictly monotone lap $u \in X_\epsilon(\mathbf{u}) = \{u \in X(\mathbf{u}) \mid J_E[u] < \mathcal{J}_E + \epsilon\}$ whose length τ is at most T.*

First we need a simple lemma.

Lemma 4.8. *Let $B > A > 0$ and $u \in H^2([a, b])$ with $u'(a) = A$, $u'(b) = B$, $u'(t) \geqslant A$ on $[a, b]$, and $\|u''\|_{L^2} \leqslant C$. Then $|u(b) - u(a)| \geqslant A(A - B)^2/C^2$.*

Proof. This is a consequence of Hölder's inequality and the Mean Value Theorem. □

Proof of Theorem 4.7. Choose $\rho > 0$ and $\delta_2 > 0$ from property (P1). Choose $\delta_1 > 0$ as in Section 3, and choose $\delta_0 > 0$ and $\tau_0 > 0$ from Theorems 3.2 and 3.3. Finally choose $\delta_3 > 0$ so that for any $0 < \delta < \delta_3$ the minimizer from Theorem 3.2 with boundary conditions less than δ will have values and derivative values less than $C\delta < \delta_1$. Recall that we consider the case of an increasing lap so that $u'(t) > 0$ on $[0, \tau]$.

Claim. *There exist $\delta_4 > 0$ and $M > 0$ such that if $u \in X_\epsilon(\mathbf{u})$ for any $\epsilon < 1$ and $I = [a, b]$ is any interval with $\|(u|_I, u'|_I) - (u_*, 0)\|_\infty < \delta_1$ and $\|(u(a) - u_*, u'(a))\| \|(u(b) - u_*, u'(b))\| < \delta_4$, then $\|u\|_{H^2(I)} \leqslant M$.*

Proof. As in Section 3, by the C^1-bound on the values of $u|_I$ we have $J_E[u] \geqslant I_2[u]$ as defined in Step 2 of Theorem 3.2. Moreover, from Lemma 3.1 there exists $\hat{\epsilon} > 0$ such that $J_E[u] \geqslant I_2[u] \geqslant \hat{\epsilon}\|u\|^2_{H^2(I)} - (\hat{\epsilon} - \frac{1}{2}\beta)[(u - u_*)u']^b_a$. Thus $\delta_4 > 0$ can be chosen small enough such that $|(\hat{\epsilon} - \frac{1}{2}\beta)[(u - u_*)u']^b_a| < 1$ which implies that $\|u\|^2_{H^2(I)} \leqslant (\mathcal{J}_E + 2)/\hat{\epsilon} = M^2$. □

Let $\gamma = \min\{\delta_0, \delta_1, \delta_2, \delta_3, \delta_4\}$. Now we show that there exists $l > 0$ such that for any $\epsilon < 1$, $u \in X_\epsilon(\mathbf{u})$ and any interval I containing $t(u_*)$ with $u|_I(t) \in [u_* - l, u_* + l]$ for all $t \in I$ and $u'(t(u_*)) < \gamma/2$, we have $\|(u|_I, u'|_I) - (u_*, 0)\|_\infty < \delta_1$. Suppose $u'(t(u_*)) < \gamma/2$. Let q_1 be the smallest time in $[0, t(u_*)]$ such that $u'(t) \leqslant \gamma$ for all $t \in [q_1, t(u_*)]$. Also let q_2 be the largest time in $[t(u_*), \tau]$ such that $u'(t) \leqslant \gamma$ for all $t \in [t(u_*), q_2]$. If $q_1 = 0$, then $u_* - u(q_1) = u_* - u_1$. If $q_1 > 0$ then $u'(q_1) = \gamma$, and there exists $p_1 \in (q_1, t(u_*))$ such that $u'(p_1) = 3\gamma/4$. Using Lemma 4.8 and the above claim, we have $u(p_1) - u(q_1) \geqslant C\gamma^3$ with constant independent of u, which implies that $u_* - u(q_1) \geqslant C\gamma^3$. Using the same arguments $u(q_2) - u_* = u_2 - u_*$ or $u(q_2) - u_* \geqslant C\gamma^3$. Let $l = \min\{u_* - u_1, u_2 - u_*, C\gamma^3, \gamma\}$. Then $[u_* - l, u_* + l]$ has the property that if $u|_I(t) \subset [u_* - l, u_* + l]$ then $\|(u|_I, u'|_I) - (u_*, 0)\|_\infty < \gamma \leqslant \delta_1$.

We are now ready to prove the theorem. Fix $0 < \epsilon < 1$. Choose $\delta < l$ such that $C\delta < l$ where the C is the constant from Theorem 3.2. Let $u \in X_\epsilon(\mathbf{u})$.

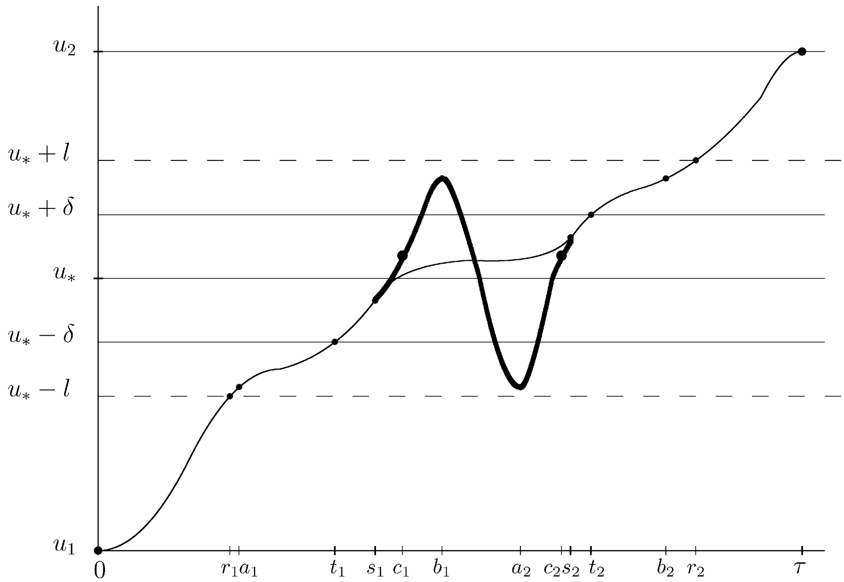

Fig. 4. Illustration of Case 1 in the proof of Theorem 4.7. The thin line represents the original lap u. The thick line from s_1 to s_2 represents the minimizer from Theorem 3.2 which must oscillate around u_* by Theorem 3.3.

Case 1. $u'(t(u_*)) \leqslant \delta/2$. See Fig. 4.

Let $t_1, t_2 > 0$ be such that $u(t_1) = u_* - \delta$ and $u(t_2) = u_* + \delta$. Let s_1 be the smallest time in $[t_1, t(u_*)]$ such that $u'(s_1) \leqslant \delta$, and let s_2 be the largest time in $[t(u_*), t_2]$ such that $u'(s_2) \leqslant \delta$. If $t_1 < s_1$, then $u(s_1) - u(t_1) \leqslant l$ and $|u'(t)| \geqslant \delta$ for $t \in [t_1, s_1]$ so by the Mean Value Theorem $s_1 - t_1 \leqslant l/\delta$. Similarly $t_2 - s_2 \leqslant l/\delta$. Moreover by Lemma 4.6 there exists $T_* > 0$ such that $t_1 \leqslant T_*$ and $\tau - t_2 \leqslant T_*$. Thus if $s_2 - s_1 < \max\{1, 4\tau_0\}$ where τ_0 is the constant from Theorem 3.3, then $\tau \leqslant 2T_* + 2l/\delta + \max\{1, 4\tau_0\}$. So we now consider the remaining possibility that $s_2 - s_1 \geqslant \max\{1, 4\tau_0\}$.

Since $\|(u(s_i) - u_*, u'(s_i))\| \leqslant \delta$, $i = 1, 2$, and $\|(u(t) - u_*, u'(t))\| \leqslant \delta_1$ for $t \in [s_1, s_2]$, we can use Theorem 3.2 to replace $u|_{[s_1, s_2]}$ by the minimizer of J_E over this interval to obtain a function v such that $J_E[v] < J_E[u]$. This inequality is strict since $s_2 - s_1 \geqslant 4\tau_0$ implies that the unique minimizer v is not monotone but u is monotone. Further we can make an arbitrarily small perturbation of v in $H^2([s_1, s_2])$ so to assume that v is a Morse function, i.e. v has isolated nondegenerate critical points, with $J_E[v] < J_E[u]$. Note that the choice of δ insures that $\|(v(t) - u_*, v'(t))\| \leqslant l \leqslant \delta_1$ for $t \in [s_1, s_2]$, and in fact $\|(v(t) - u_*, v'(t))\| \leqslant \delta_1$ for all $t \in [r_1, r_2]$ where r_1, r_2 are the times such that $v(r_1) = u_* - l$ and $v(r_2) = u_* + l$. Furthermore, by the choice of δ_1 and Lemma 3.1, over any interval $I = [a, b] \subset [r_1, r_2]$ with $v(a) = v(b)$ and $v'(a) = v'(b)$ we have $J_E[v|_I] \geqslant 0$. Thus such intervals can be clipped from v and decrease the action.

Let b_1 be the time where $v(b_1)$ is the first relative maximum to the right of s_1 such that $v(b_1) > u_*$. We can assume that v is monotone over $[s_1, b_1]$, because if not we can clip to make v monotone on this interval. Let a_2 be the time where $v(a_2)$ is the first relative minimum to the left of s_2 such that $v(a_2) < u_*$. Similarly, we can assume that v is monotone over $[a_2, s_2]$. The times b_1 and a_2 exist by Theorem 3.3 since $s_2 - s_1 > 4\tau_0$. Let a_1 be the point in the interval $[r_1, b_1]$

such that $v(a_1) = v(a_2)$. Let b_2 be the point in the interval $[a_2, r_2]$ such that $v(b_2) = v(b_1)$. By the clipping lemma there exist $c_1 \in [a_1, b_1]$ and $c_2 \in [a_2, b_2]$ such that $v(c_1) = v(c_2)$ and $v'(c_1) = v'(c_2)$. Let w be the function obtained from v by clipping out $v|_{(c_1,c_2)}$. Then $J[w] < J[v]$, and w is monotone. By Theorem 3.3 we have $t(v(c_1)) - s_1 < 2\tau_0$ and $s_2 - t(v(c_2)) < 2\tau_0$. Therefore we have a bound on the time of w given by of $\tau \leqslant 2T_* + 2l/\delta + 4\tau_0$.

Case 2. $u'(t(u_*)) > \delta/2$.

Let $[s_1, s_2]$ be the largest interval containing $t(u_*)$ on which $u' \geqslant \delta/4$ so that $u'(s_1) = u'(s_2) = \delta/4$ and $s_1 < t(u_*) < s_2$. By the Mean Value Theorem, the length of the interval $[s_1, s_2]$ is bounded by $4(u_2 - u_1)/\delta$. From Lemmas 4.5 and 4.8 we have $u_* - u(s_1), u(s_2) - u_* \geqslant C\delta^3$. By Lemma 4.6 there exists $T_* > 0$ such that $t(u_* - C\delta^3)$ is bounded by T_* and the length of $[t(u_* + C\delta^3), \tau]$ is bounded by T_*. Therefore $\tau \leqslant 2T_* + 4(u_2 - u_1)/\delta$.

Let $T > 0$ be the maximum time bound in the above cases. For each $\epsilon > 0$, we have constructed $w \in X(\mathbf{u})$ such that $J_E[w] < \mathcal{J} + \epsilon$ and $\tau \leqslant T$. $\square$

The following two lemmas are proved in [11]. For $\tau > 0$ and $a \in \mathbb{R}$, let $J^a_{\tau,E}(\mathbf{u})$ denote the sublevel set $\{u \in X_\tau(\mathbf{u}) \mid J_E[u] \leqslant a\}$.

Lemma 4.9. *(See Lemma 3.5 in [11].) There exists $C(\tau, a, \mathbf{u})$ such that $\|u\|_{H^2([0,\tau])} \leqslant C$ for all $u \in J^a_{\tau,E}(\mathbf{u})$.*

Lemma 4.10. *(See Lemma 3.6 in [11].) Suppose $u_n \in X(\mathbf{u})$ with both $\|u_n\|_{H^2([0,\tau_n])}$ and τ_n uniformly bounded. Then there exists a subsequence u_{n_k} such that $u_{n_k} \rightharpoonup u$ in $H^2([0, \tau])$ and $\liminf_{n_k\to\infty} J_E[u_{n_k}] \geqslant J_E[u]$.*

Theorem 4.11. *There exists a minimizing sequence which converges weakly in $H^2([0, \tau])$.*

Proof. By Theorem 4.7 we can choose u_n so that $J_E[u_n] < \mathcal{J}_E + 1/n$ and $\tau_n \leqslant T$. By Lemma 4.10, J_E is sequentially weakly lower semicontinuous and coercive along a subsequence u_{n_k}. By standard theory $u_{n_k} \rightharpoonup u \in H^2([0, \tau])$ for some $\tau < \infty$ such that $J_E[u] = \mathcal{J}_E(\mathbf{u})$. $\square$

Since weak convergence in H^2 implies strong convergence in C^1_{loc}, the function u obtained from Theorem 4.11 will be monotone but not necessarily in $X(\mathbf{u})$, since it is possible that it has critical inflection points. For notation purposes let $\mathrm{cl}\, X(\mathbf{u}) = \{u \in H^2([0, \tau])$ for some $\tau > 0 \mid \tau_n \to \tau$ and $u_n \rightharpoonup u$ for some $u_n \in X_{\tau_n}(\mathbf{u})\}$. To rule out critical inflection points, we need the following lemma from [11].

Lemma 4.12. *(See Lemma 3.10 in [11].) Suppose $[w_1, w_2] \subset [u_1, u_2]$ with $u_* \notin [w_1, w_2]$ and $u \in \mathrm{cl}\, X(\mathbf{u})$. Assume $0 < u'(t(w_1)) = u'(t(w_2)) = b < \delta_0$, where δ_0 satisfies property* (P1). *If $u'' \not\equiv 0$, then $J_E[w] < J_E[u|_{[t(w_1),t(w_2)]}]$ where $w(t) = bt + w_1$ on $[0, (w_2 - w_1)/b]$.*

Theorem 4.13. *If $u \in \mathrm{cl}\, X(\mathbf{u})$ is a minimizer of J_E then $u \in X(\mathbf{u})$.*

Proof. First suppose u has a critical inflection point at t_0, where $u(t_0) \neq u_*$. Since u is monotone, t_0 is contained in some maximal compact interval of critical points I. By continuity, for every $b \in \mathbb{R}$ with $|b|$ sufficiently small, there is an interval $[t_1, t_2]$ containing I such that $u'(t_1) = u'(t_2) = b$. Let $w_1 = u(t_1)$ and $w_2 = u(t_2)$. Then using Lemma 4.12 we can construct a function w such that $J_E[w] < J_E[u|_{[t_1,t_2]}]$. Replacing $u|_{[t_1,t_2]}$ by w yields a function $\hat{u} \in H^2([0, \hat{\tau}])$ such that $J_E[\hat{u}] < J_E[u]$, which contradicts the fact that u is a minimizer. Therefore u has no critical inflection points except possibly at u_*.

Next suppose $(u(t_0), u'(t_0)) = (u_*, 0)$. Then t_0 is contained in a maximal compact interval of critical points I. If $|I| < \max\{1, 4\tau_0\}$, we can insert an interval on which the function is identically u_* without changing the action since the integrand of J_E is zero at $(u_*, 0)$. Thus we can assume $|I| \geqslant \max\{1, 4\tau_0\}$. Choose $\delta_1 > 0$ and $\delta > 0$ as in Theorem 4.7. Then we can use the same arguments of inserting the minimizer and clipping as we did in Case 2 in the proof of Theorem 4.7. The result is a function $w \in X(\mathbf{u})$ such that $J_E[w] < J_E[u]$ which again contradicts the assumption that u is a minimizer. Therefore u has no critical inflection points and $u \in X(\mathbf{u})$. □

Corollary 4.14. *Suppose $u_* \in (u_1, u_2)$ is a saddle-focus and the only critical point of H in $[u_1, u_2] \subset I$, an interval component of J_E. Then $X(\mathbf{u})$ contains a minimizer of J_E.*

Proof. Theorems 4.7, 4.11, 4.13 imply the result. □

4.4. Multiple critical points

The arguments and results of the previous section extend immediately to the case of multiple critical points.

Theorem 4.15. *Suppose $u_1^*, u_2^*, \dots, u_n^* \in (u_1, u_2)$ are saddle-foci and are the only critical points of H in $[u_1, u_2] \subset I$, and interval component of J_E. Then $X(\mathbf{u})$ contains a minimizer of J_E.*

Proof. We must show that the arguments in the proof of Theorem 4.7 apply in this situation, i.e. there exists $T > 0$ such that for every $\epsilon > 0$ there exists a strictly monotone function $u \in X(\mathbf{u})$ such that $J_E[u] < \mathcal{J}_E + \epsilon$ and $\tau \leqslant T$. Let $l = \min\{l_j, |u_{j+1}^* - u_j^*|/2 \mid j = 1, \dots, n-1\}$ where l_j is chosen for u_j^* as in the proof of Theorem 4.7. Choose $\delta < l$ such that $C\delta < l$ where C is the constant of Theorem 3.2. Let $u \in X(\mathbf{u})$ such that $J_E[u] < \mathcal{J}_E + \epsilon$. Outside of the intervals $[u_j^* - l, u_j^* + l]$ we have a finite number of closed intervals that contain no critical points. Therefore by Lemma 4.6 there exists a time T_j for each of these $n + 1$ closed intervals such that the time spent in the jth interval is bounded by T_j. Within $[u_j^* - l, u_j^* + l]$ we can alter u to obtain w with the time spent in $[u_j^* - l, u_j^* + l]$ bounded by a constant depending on $\tau_0, l, 1/\delta$, and $u_2 - u_1$ as in Theorem 4.7. Theorems 4.11 and 4.13 now imply the result. □

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